Baxter's T-Q Relation and Bethe Ansatz of Discrete Quantum Pendulum and Sine-Gordon Model

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Abstract

Using the Baxter's T-Q relation derived from the transfer matrix technique, we consider the diagonalization problem of discrete quantum pendulum and discrete quantum sine-Gordon Hamiltonian from the algebraic geometry aspect. For a finite chain system of the size L, when the spectral curve degenerates into rational curves, we have reduced the Baxter's T-Q relation into a polynomial equation; the connection of T-Q polynomial equation with the algebraic Bethe Ansatz is clearly established . In particular, for $L = 4$ it is the case of rational spectral curves for the discrete quantum pendulum and discrete sine-Gordon model. To these Baxter's T-Q polynomial equations, we have obtained the complete and explicit solutions with a detailed understinding of the quantitative and qualitative structure of solutions. In general the model possesses a spectral curve with a generic parameter, we have conducted certain qualitative study on the algebraic geometry of this high genus Riemann surface incorporating the Baxter's T-Q relation.

1 Introduction

In the early seventies, R. Baxter proposed the method of Q-operator and the T-Q relation in his renowned solution of the 8-vertex model and the spin $\frac{1}{2}$ XYZ chain in soluble statistical mechanics [2, 3]. Since then, the method has played a powerful mechanism till nowadays in the 2-dimensional exactly solvable lattice models and the corresponding quantum spin-chain Hamiltonians. The method of quantum inverse scattering / algebraic Bethe Ansatz developed by the Lenningrad school in the early eighties [7, 11] systematized earlier results on 2-dimensional integrable lattice models, and paved the way for the far-reaching effects in both mathematical and physical development in the past two decades. Within the framework of quantum inverse scattering method, Izergin and Korepin, also Tarasov [13] found the $\mathcal{L}$-operator with $\mathbf{C}^N$-operators entries for the massless lattice sine-Gordon model, which satisfies the Yang-Baxter equation for the R-matrix of XXZ model with the anisotropy parameter $\frac{1}{2}(q+q^{-1}), q^N = 1$, (see the formula (4) in the content of this paper). A slightly modified version of that $\mathcal{L}$-operator did appear also in the study of chiral Potts N-state model [4]. On the other hand for a fixed finite size L of the system while N varying, certain Hamiltonians of physical interest have again presented a intimate relationship with the transfer matrix in the above theory. For L=3, the Hamiltonian, first proposed by Faddeev and Kashaev [8] then investigated in a rigorously mathematical manner in our previous article [14], has shown an intimate connection with the Hofstadter model [1, 9, 15], a renowned Bloch system with a constant external magnetic field. The quantum inverse scattering method allows one to calculate the spectrum of the Hamiltonian by solving the (algebraic) Bethe Ansatz. In [14] we formulate the method through the Baxter's T-Q relation on the spectral curve via the Baxter's vacuum state [3, 5] from the algebraic geometry aspect. In addition, a general scheme of diagonalizing the transfer matrix for a finite size L by means of the Baxter's T-Q relation (or the Bethe equation) on the spectral curve has also been discovered. Though it is rather difficult now to extract explicit quantitative information for the spectrum problem by this approach due to the complicated functional theory of the high genus spectral curve, we demonstrate in this paper that the polynomial equation derived from the Baxter's T-Q relation while the curve degenerates into rational curves is indeed equivalent to the usual Bethe Ansatz in the physical literature; the general form of transfer matrix in L=4 gives rise to the discrete quantum pendulum and the discrete quantum sine-Gordon model (the SG model) proposed in [6, 10].

In this article we make a thorough study of the discrete quantum pendulum and the SG model in the formalism of Baxter's T-Q relation through the transfer matrix technique, (for the Hamiltonians, see (20) (21) of Sect. 2). In our approach, all the considerations are made in the context of Hamiltonian chains of a fixed finite size L, and the mathematical treatment takes advantage of special features only presented in L=4. For the general spectral curve upon which the Baxter's T-Q relation is formulated, the genus is high with the order of the fifth power of N. The analysis in algebraic geometry of this family of curves as cover of elliptic curves has been made, in hope that the elliptic function theory would eventually play a role in solutions of the T-Q relation. For the case of rational spectral curves, though the geometry of the spectral curve turns to a trivial one, the determination of the solutions inevitably requires the subtle analysis of Baxter vacuum state to extract the essential data for polynomials involved in the equation, then carry out the necessary algebraic study of a certain "over-determined" system of q-difference equations for a root of unity $q^N = 1$, which is still a difficult problem for an arbitrary finite size L. For L=4, by taking the special symmetric algebraic structure of polynomial functions into account we are able to obtain the explicit solution of the Baxter's T-Q polynomial relation for the discrete quantum pendulum and the SG model. The result is complete from both the quantitative and qualitative aspects, and provides a sound mathematical treatment on the problem raised before in [6, 10]. Our method to the eigenvalue problem of these Hamiltonians is, also as shown in [14] for L=3 on the Hofstadter model, not only more fundamental, but also mathematically tractable than the usual Bethe Ansatz technique. It appears that a certain mathematical theory of q-Strum-Liouville type problem would entangle with this type of Baxter's T-Q polynomial equation arising from physical problems.

This paper is organized as follows. In Sect. 2, we review some basic construction of the transfer matrix and the Baxter's T-Q relation in context of quantum inverse scattering method, with the spectral data in a high genus algebraic curve depending on the size L and the parameters which are involved. Some formulation in [14] will be recalled here for the sake of completeness. Then we introduce the constraint of the parameters of the spectral curve for the discussion of discrete quantum pendulum and the SG model. In Sect. 3, we discuss a canonical procedure of reducing the Baxter's T-Q relation to a polynomial equation when the spectral curve is degenerated to rational curves for a general finite size L. By converting the parameters to one special case which has been studied in our earlier paper [14], we obtain the Baxter's T-Q relation in a polynomial form. In Sect. 4 we apply the result of the previous section to the case L=4 incorporating with the parameter's constraint of the discrete quantum pendulum and SG model. The symmetric Baxter's T-Q polynomial relation is introduced with a general discussion on the qualitative nature of its solutions. In Sect. 5 we explicitly construct the complete solution of symmetric Baxter's T-Q polynomial relation, among which the rational degenerated cases of the discrete quantum pendulum and SG model are included. Both the quantitative and qualitative nature of solutions are revealed in the process of this mathematical derivation, and these solutions recover the Bethe Ansatz in physical literature that were previously defined using *ad hoc* argument. In Sect. 6 we consider the discrete quantum pendulum and the SG model with a general spectral curve, a Riemann surface with a very high genus. We conduct a geometric study of the curve, and discover a canonical induced family of elliptic curve over which the general spectral curves lie as branched covers. A primitive analysis on the relation of the geometry of these curves with the eigenvalue problem of the physical models involved is given from the qualitative aspect through the Baxter's T-Q relation. In Sect. 7 we present the conclusion remark with a discussion of our future problems related to the Baxter's T-Q relation. We end with the appendix of presenting a detailed identification of the sine-Gordon integral in [10] and the one given in this paper.

Notations. To present our work, we prepare some notations. In this paper, $\mathbf{Z}, \mathbf{R}$, $\mathbf{C}$ will denote the ring of integers, real, complex numbers respectively, $\mathbf{N} = \mathbf{Z}_{>0}$, $\mathbf{Z}_N = \mathbf{Z}/N\mathbf{Z}$, and $\mathrm{i} = \sqrt{-1}$. For a positive integers n, we denote $\stackrel{n}{\otimes} \mathbf{C}^N$ the tensor product of n-copies of the vector space $\mathbf{C}^N$. We use the notation of q-shifted factorials,

$$(a;\mathsf{q})_0 = 1, \quad (a;\mathsf{q})_n = (1-a)(1-a\mathsf{q})\cdots(1-a\mathsf{q}^{n-1}), \quad n \in \mathbf{N} .$$

We shall minimize the repetition of materials from our previous article [14], and so opt to use the same notation conventions as much as possible.

2 Transfer Matrix, Baxter Vacuum State and T-Q Equation

In this section we first recall some formulae in quantum inverse scattering method which we shall need in this paper. Most of this material can be found in [14] including the original references. After that, we specify our discussion on the case which leads to the discrete quantum pendulum and the SG Hamiltonian, the models we mainly concern in this work.

In this paper, N will always denote an odd positive integer with $M = [\frac{N}{2}]$,

$$N = 2M+1 , \quad M \geq 1 ,$$

and ω is a primitive N-th root of unity, $q := \omega^{\frac{1}{2}}$ with $q^N = 1$, i.e., $q = \omega^{M+1}$. An element v in the vector space $\mathbf{C}^N$ is represented by a sequence of coordinates, $(v_k \mid k \in \mathbf{Z})$, with the N-periodic condition, $v_k = v_{k+N}$, equivalently to say, $v = (v_k)_{k \in \mathbf{Z}_N}$. The standard basis of $\mathbf{C}^N$ will be denoted by $|k\rangle$, with the dual basis of $\mathbf{C}^{N*}$ by $\langle k|$ for $k \in \mathbf{Z}_N$.

Let Z, X be the generators of the Weyl algebra with the following commutation relation and the N-th power identity,

$$ZX = \omega XZ, \quad Z^N = X^N = I \ , \tag{1}$$

and denote $Y := ZX$. Then

$$XY = \omega^{-1} YX \ , \quad YZ = \omega^{-1} ZY \ , \quad Y^N = 1 \ .$$

The canonical representation of the Weyl algebra is the unique irreducible one on $\mathbf{C}^N$ with the expression:

$$Z(v)_k = \omega^n v_k \ , \quad X(v)_k = v_{k-1} \ , \quad Y(v)_k = \omega^k v_{k-1} \quad \text{for} \quad v = (v_k) \in \mathbf{C}^N.$$

By using the above operators, we consider a solution of the Yang-Baxter equation for a slightly modified R-matrix of the XXZ-model, appeared first in [8], then studied in more details in [14]. The L-operator is given by the following 2×2 matrix of the operator-valued entries acting on the quantum space $\mathbf{C}^N$ with the parameter $h = [a : b : c : d]$ in the projective 3-space $\mathbf{P}^3$,

$$L_h(x) = \begin{pmatrix} aY & xbX \\ xcZ & d \end{pmatrix} \ , \quad x \in \mathbf{C} \ , \tag{2}$$

which satisfies the Yang-Baxter relation

$$R(x/x')(L_h(x) \bigotimes_{aux} 1)(1 \bigotimes_{aux} L_h(x')) = (1 \bigotimes_{aux} L_h(x'))(L_h(x) \bigotimes_{aux} 1) R(x/x') \ , \tag{3}$$

where the script letter "aux" indicates an operation taking on the auxiliary space $\mathbf{C}^2$, $R(x)$ is the matrix of a 2-tensor of the auxiliary space with the following numerical expression,

$$R(x) = \begin{pmatrix} x\omega - x^{-1} & 0 & 0 & 0 \\ 0 & \omega(x - x^{-1}) & \omega - 1 & 0 \\ 0 & \omega - 1 & x - x^{-1} & 0 \\ 0 & 0 & 0 & x\omega - x^{-1} \end{pmatrix} .$$

The operator (2) is related to the following one in [13] on the study of sine-Gordon lattice model using the R-matrix of XXZ-model and the Weyl operators U, V: $UV = q^{-1}VU$, $U^N = V^N = 1$,

$$L_h^*(x) = \begin{pmatrix} aqU & xbV^{-1} \\ xcV & dU^{-1} \end{pmatrix} \ , \quad \mathsf{R}(x) = \begin{pmatrix} xq - x^{-1}q^{-1} & 0 & 0 & 0 \\ 0 & x - x^{-1} & q - q^{-1} & 0 \\ 0 & q - q^{-1} & x - x^{-1} & 0 \\ 0 & 0 & 0 & xq - x^{-1}q^{-1} \end{pmatrix} , \tag{4}$$

which satisfy the Yang-Baxter relation

$$\mathsf{R}(x/x')((L_h^*(x) \bigotimes_{aux} 1) \otimes (1 \bigotimes_{aux} L_h^*(x')) = ((1 \bigotimes_{aux} L_h^*(x') \otimes (L_h^*(x) \bigotimes_{aux} 1)) \mathsf{R}(x/x') \ .$$

By the identification

$$Z = VU, \quad X = V^{-1}U,$$

which implies $Y = qU^2$, (or equivalently $U = q^{\frac{-1}{2}} Y^{\frac{1}{2}}, V = q^{\frac{1}{2}} Z Y^{\frac{-1}{2}}$), the explicit connection between $L_h(x)$ of (2) and $L_h^*(x)$ of (4) is given by

$$L_h^*(x) = L_h(x) Y^{\frac{-1}{2}} q^{\frac{1}{2}}. \tag{5}$$

By the matrix-product on auxiliary spaces and tensor-product of quantum spaces, the L-operator for a finite size L with the the period boundary condition and the parameter $\vec{h} = (h_0, \ldots, h_{L-1}) \in (\mathbf{P}^3)^L$,

$$L_{\vec{h}}(x) = \bigotimes_{j=0}^{L-1} L_{h_j}(x) \quad (:= L_{h_0}(x) \otimes L_{h_1}(x) \otimes \ldots \otimes L_{h_{L-1}}(x)) \ ,$$

again satisfies the relation (3), hence it gives rise to the commuting family of transfer matrices

$$T_{\vec{h}}(x) = \mathrm{tr}_{aux}(L_{\vec{h}}(x)) \ , \quad x \in \mathbf{C} \ . \tag{6}$$

The same conclusion holds for $L_{\vec{h}}^*(x), T_{\vec{h}}^*(x)$ defined by

$$L_{\vec{h}}^*(x) = \bigotimes_{j=0}^{L-1} L_{h_j}^*(x) \ , \quad T_{\vec{h}}^*(x) = \mathrm{tr}_{aux}(L_{\vec{h}}^*(x)),$$

and by (5), the connection between these two families of transfer matrices is given by

$$T_{\vec{h}}^*(x) = T_{\vec{h}}(x) D^{\frac{-1}{2}} \ , \quad \text{where} \quad D := q^{-L} \overset{L}{\otimes} Y. \tag{7}$$

We now summarize some basic facts on Bethe equation and the Baxter vacuum state on the spectral curve in the diagonalization problem of the transfer matrix $T_{\vec{h}}(x)$, (for the details, see [14]). In computing the spectra of $T_{\vec{h}}(x)$, one can apply the gauge transform technique on $L_{h_j}(x)$ in the following form,

$$\widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = A_j L_{h_j}(x) A_{j+1}^{-1} \ , \quad A_j = \begin{pmatrix} 1 & \xi_j - 1 \\ 1 & \xi_j \end{pmatrix} \quad 0 \leq j \leq L-1,$$

with $A_L := A_0$. We have

$$\widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} F_{h_j}(x, \xi_j - 1, \xi_{j+1}) & -F_{h_j}(x, \xi_j - 1, \xi_{j+1} - 1) \\ F_{h_j}(x, \xi_j, \xi_{j+1}) & -F_{h_j}(x, \xi_j, \xi_{j+1} - 1) \end{pmatrix} \ ,$$

where $F_h(x, \xi, \xi') := \xi' a Y - x b X + \xi' \xi x c Z - \xi d$. Accordingly, for $\vec{h} \in (\mathbf{P}^3)^L$ and $\vec{\xi} = (\xi_0, \ldots, \xi_{L-1}) \in (\mathbf{C}^N)^L$, the modified L-operator becomes

$$\widetilde{L}_{\vec{h}}(x, \vec{\xi}) := \bigotimes_{j=0}^{L-1} \widetilde{L}_{h_j}(x, \xi_j, \xi_{j+1}) = \begin{pmatrix} \widetilde{L}_{\vec{h};11}(x, \vec{\xi}) & \widetilde{L}_{\vec{h};12}(x, \vec{\xi}) \\ \widetilde{L}_{\vec{h};21}(x, \vec{\xi}) & \widetilde{L}_{\vec{h};22}(x, \vec{\xi}) \end{pmatrix}, \quad \xi_L := \xi_0 \ .$$

As the procedure of gauge transform keeps the same trace, we have $T_{\vec{h}}(x) = \mathrm{tr}_{aux}(\widetilde{L}_{\vec{h}}(x, \vec{\xi}))$. For a given $\vec{h}$, we will consider the variable $(x, \vec{\xi})$ lies only on the curve $\mathcal{C}_{\vec{h}}$ defined by the system of equations,

$$\mathcal{C}_{\vec{h}} : \quad \xi_j^N = (-1)^N \frac{\xi_{j+1}^N a_j^N - x^N b_j^N}{\xi_{j+1}^N x^N c_j^N - d_j^N} \quad , \qquad j = 0, \ldots, L-1, \tag{8}$$

which will be called the spectral curve in this paper. The Baxter vacuum state[1] over $\mathcal{C}_{\vec{h}}$ is the family of vectors $|p\rangle \in \stackrel{L}{\otimes} \mathbf{C}^N$ with the form

$$|p\rangle := |p_0\rangle \otimes \ldots \otimes |p_{L-1}\rangle \in \stackrel{L}{\otimes} \mathbf{C}^N , \quad p \in \mathcal{C}_{\vec{h}} ,$$

where $|p_j\rangle$ is the vector in $\mathbf{C}^N$ governed by the relation,

$$\langle 0|p_j\rangle = 1 , \quad \frac{\langle m|p_j\rangle}{\langle m-1|p_j\rangle} = \frac{\xi_{j+1}a_j\omega^m - xb_j}{-\xi_j(\xi_{j+1}xc_j\omega^m - d_j)} . \tag{9}$$

The constraint of $(x, \vec{\xi})$ on the curve $\mathcal{C}_{\vec{h}}$ ensures that the following properties hold for the Baxter vacuum state,

$$\widetilde{L}_{\vec{h};11}(x,\vec{\xi})|p\rangle = |\tau_- p\rangle\Delta_-(p), \quad \widetilde{L}_{\vec{h};22}(x,\vec{\xi})|p\rangle = |\tau_+ p\rangle\Delta_+(p) , \quad \widetilde{L}_{\vec{h};21}(x,\vec{\xi})|p\rangle = 0 ,$$

where $\Delta_\pm, \tau_\pm$ are (rational) functions and automorphisms of $\mathcal{C}_{\vec{h}}$ defined by

$$\begin{aligned}
\Delta_-(x,\xi_0,\ldots,\xi_{L-1}) &= \prod_{j=0}^{L-1}(d_j - x\xi_{j+1}c_j) , \\
\Delta_+(x,\xi_0,\ldots,\xi_{L-1}) &= \prod_{j=0}^{L-1}\frac{\xi_j(a_jd_j - x^2b_jc_j)}{\xi_{j+1}a_j - xb_j} , \\
\tau_\pm : (x,\xi_0,\ldots,\xi_{L-1}) &\mapsto (q^{\pm 1}x, q^{-1}\xi_0,\ldots,q^{-1}\xi_{L-1}) .
\end{aligned} \tag{10}$$

This implies that under the action of the transfer matrix, the Baxter vacuum state is decoupled as the sum of those under $\tau_\pm$:

$$T_{\vec{h}}(x)|p\rangle = |\tau_- p\rangle\Delta_-(p) + |\tau_+ p\rangle\Delta_+(p) , \quad \text{for} \ \ p \in \mathcal{C}_{\vec{h}} . \tag{11}$$

For a common eigenvector $\langle\varphi| \in \stackrel{L}{\otimes} \mathbf{C}^{N*}$ of the transfer matrices $T_{\vec{h}}(x)$, the eigenvalue $\Lambda(x)$ is a polynomial of x, i.e., $\Lambda(x) \in \mathbf{C}[x]$. The function $Q(p)$ on $\mathcal{C}_{\vec{h}}$ defined by $Q(p) := \langle\varphi|p\rangle$ for $p \in \mathcal{C}_{\vec{h}}$ satisfies the following equation, which will be called the Baxter's T-Q relation (or the Bethe equation) for $T_{\vec{h}}(x)$,

$$\Lambda(x)Q(p) = \Delta_-(p)Q(\tau_-(p)) + \Delta_+(p)Q(\tau_+(p)) , \quad \text{for } p \in \mathcal{C}_{\vec{h}} . \tag{12}$$

Note that the $\langle\varphi|$ is again a common eigenvector of $T^*_{\vec{h}}(x)$ with the eigenvalue $\Lambda^*(x) = q^n\Lambda(x)$, where q^n is eigenvalue of $D^{\frac{-1}{2}}$ for $\langle\varphi|$. The Baxter's T-Q relation for $T^*_{\vec{h}}(x)$ becomes

$$\Lambda^*(x)Q(p) = \Delta^*_-(p)Q(\tau_-(p)) + \Delta_+(p)^*Q(\tau_+(p)) , \quad \text{for } p \in \mathcal{C}_{\vec{h}} . \tag{13}$$

where $\Lambda^*(x) = q^n\Lambda(x)$, $\Delta^*_-(p) = q^n\Delta_-(p)$, $\Delta_+(p)^* = q^n\Delta_+(p)$ for $n \in \mathbf{Z}_N$. For L=4, we have

$$T_{\vec{h}}(x) = \mathrm{T}_0 + x^2\mathrm{T}_2 + x^4\mathrm{T}_4 , \qquad T^*_{\vec{h}}(x) = \mathrm{T}^*_0 + x^2\mathrm{T}^*_2 + x^4\mathrm{T}^*_4 . \tag{14}$$

The T_{2j}s are operators of $\stackrel{4}{\otimes} \mathbf{C}^N$ with the expressions,

$$\begin{aligned}
\mathrm{T}_0 &= a_0a_1a_2a_3\omega^2 D + d_0d_1d_2d_3 , \\
\mathrm{T}_2 &= a_0a_1b_2c_3 Y\otimes Y\otimes X\otimes Z + b_0c_1a_2a_3 X\otimes Z\otimes Y\otimes Y + a_0b_1c_2a_3 Y\otimes X\otimes Z\otimes Y \\
&\quad + a_0b_1d_2c_3 Y\otimes X\otimes 1\otimes Z + b_0d_1c_2a_3 X\otimes 1\otimes Z\otimes Y + b_0d_1d_2c_3 X\otimes 1\otimes 1\otimes Z \\
&\quad + (a_jY \leftrightarrow d_j, b_jX \leftrightarrow c_jZ) , \\
\mathrm{T}_4 &= b_0c_1b_2c_3 DC^{-1} + c_0b_1c_2b_3 C , \quad C := Z\otimes X\otimes Z\otimes X .
\end{aligned}$$

[1]The Baxter vacuum state here was called by the Baxter vector in [8, 14]

By (14), one can also obtain the expressions of T^*_{2j}s. Note that C, D T_2, T^*_2 all commute with each other.

In this paper, we will mainly restrict ourself in the study of discrete quantum pendulum and SG model, which is on the case L=4 with the following constraint[2] on $\vec{h}$ depending on a parameter $k \in \mathbf{C}$,

$$\begin{array}{lll} a_jd_j = q^{-1}, \ b_jc_j = -k^{-1} \ , & \text{for} & j = 0, 2 \ , \\ a_jd_j = q^{-1}, \ b_jc_j = -k \ , & \text{for} & j = 1, 3 \ . \end{array} \tag{15}$$

The operators $\mathrm{T}_{2j}, \mathrm{T}^*_{2j}$ in (14) now have the following forms,

$$\begin{array}{rl} \mathrm{T}_0 = & \frac{1}{d_0d_1d_2d_3}D + d_0d_1d_2d_3 \ , \qquad\qquad \mathrm{T}_4 = \frac{c_1c_3}{k^2c_0c_2}DC^{-1} + \frac{k^2c_0c_2}{c_1c_3}C \ , \\ -\mathrm{T}_2 = & \frac{kc_0d_2d_3}{c_1}U_1 + \frac{d_0c_1d_3}{kc_2}U_2 + \frac{kd_0d_1c_2}{c_3}U_3 + \frac{d_1d_2c_3}{kc_0}U_4 + \frac{c_1}{kc_0d_2d_3}DU_1^{-1} + \frac{kc_2}{d_0c_1d_3}DU_2^{-1} \\ & + \frac{c_3}{kd_0d_1c_2}DU_3^{-1} + \frac{kc_0}{d_1d_2c_3}DU_4^{-1} + \frac{kd_0c_1}{qd_2c_3}V_1 + \frac{kqd_2c_3}{d_0c_1}DV_1^{-1} + \frac{d_3c_0}{kqd_1c_2}V_4 + \frac{qd_1c_2}{kd_3c_0}DV_4^{-1} \ ; \\ \mathrm{T}^*_0 = & \frac{1}{d_0d_1d_2d_3}D^{\frac{1}{2}} + d_0d_1d_2d_3D^{\frac{-1}{2}} \ , \qquad \mathrm{T}^*_4 = \frac{c_1c_3}{k^2c_0c_2}D^{\frac{1}{2}}C^{-1} + \frac{k^2c_0c_2}{c_1c_3}D^{\frac{-1}{2}}C \ , \\ -\mathrm{T}^*_2 = & \frac{kc_0d_2d_3}{c_1}D^{\frac{-1}{2}}U_1 + \frac{d_0c_1d_3}{kc_2}D^{\frac{-1}{2}}U_2 + \frac{kd_0d_1c_2}{c_3}D^{\frac{-1}{2}}U_3 + \frac{d_1d_2c_3}{kc_0}D^{\frac{-1}{2}}U_4 + \frac{c_1}{kc_0d_2d_3}D^{\frac{1}{2}}U_1^{-1} \\ & + \frac{kc_2}{d_0c_1d_3}D^{\frac{1}{2}}U_2^{-1} + \frac{c_3}{kd_0d_1c_2}D^{\frac{1}{2}}U_3^{-1} + \frac{kc_0}{d_1d_2c_3}D^{\frac{1}{2}}U_4^{-1} + \frac{kd_0c_1}{qd_2c_3}D^{\frac{-1}{2}}V_1 + \frac{kqd_2c_3}{d_0c_1}D^{\frac{1}{2}}V_1^{-1} \\ & + \frac{d_3c_0}{kqd_1c_2}D^{\frac{-1}{2}}V_4 + \frac{qd_1c_2}{kd_3c_0}D^{\frac{1}{2}}V_4^{-1} \ , \end{array} \tag{16}$$

where U_j, V_j are the operators defined by

$$\begin{array}{llll} U_1 = Z \otimes X \otimes 1 \otimes 1, & U_2 = 1 \otimes Z \otimes X \otimes 1, & U_3 = 1 \otimes 1 \otimes Z \otimes X, & U_4 = X \otimes 1 \otimes 1 \otimes Z, \\ V_1 = 1 \otimes Z \otimes Y \otimes X, & V_2 = X \otimes 1 \otimes Z \otimes Y, & V_3 = Y \otimes X \otimes 1 \otimes Z, & V_4 = Z \otimes Y \otimes X \otimes 1 \ . \end{array}$$

It is easy to see that the following relations hold among the above operators,

$$\begin{array}{ll} U_{j+1}U_j = \omega U_jU_{j+1} \ , & V_{j+1}V_j = \omega^2 V_jV_{j+1}, \quad (U_5 := U_1, V_5 := V_1), \\ U_iU_j = U_jU_i \ , & V_iV_j = V_jV_i \ \text{ if } i \equiv j \pmod 2; \\ V_1 = U_3U_2 \ , \ V_2 = U_4U_3, & V_3 = U_1U_4 \ , \ V_4 = U_2U_1 \ ; \\ U_1U_3 = C \ , \ U_2U_4 = C^{-1}D \ , & V_1V_3 = V_2V_4 = \omega D \ . \end{array} \tag{17}$$

By (15), the function $\Delta_\pm$ of $\mathcal{C}_{\vec{h}}$ become

$$\begin{array}{ll} \Delta_-(x, \xi_0, \ldots, \xi_3) & = \prod_{j=0}^3 (d_j - x\xi_{j+1}c_j) \ , \\ \Delta_+(x, \xi_0, \ldots, \xi_3) & = \frac{d_0d_1d_2d_3(1+x^2qk^{-1})^2(1+x^2qk)^2}{(1+x\xi_1^{-1}d_0c_0^{-1}qk^{-1})(1+x\xi_3^{-1}d_2c_2^{-1}qk^{-1})(1+x\xi_2^{-1}d_1c_1^{-1}qk)(1+x\xi_0^{-1}d_3c_3^{-1}qk)} \ . \end{array} \tag{18}$$

Using the relation (9), the Baxter vacuum state $|p\rangle = \otimes_{j=0}^3 |p_j\rangle$ have the following expression,

$$\langle m|p_j\rangle = \left\{ \begin{array}{ll} \frac{\xi_{j+1}^m q^{m^2} (-k^{-1}x\xi_{j+1}^{-1}c_j^{-1}d_jq^{-1}; \ \omega^{-1})_m}{\xi_j^m d^{2m} (x\xi_{j+1}c_jd_j^{-1}q^2; \ \omega)_m} & \text{for even } j, \\ \frac{\xi_{j+1}^m q^{m^2} (-kx\xi_{j+1}^{-1}c_j^{-1}d_jq^{-1}; \ \omega^{-1})_m}{\xi_j^m d_j^{2m} (x\xi_{j+1}c_jd_j^{-1}q^2; \ \omega)_m} & \text{for odd } j \ . \end{array} \right.$$

In this paper we shall mainly study the diagonalization problem of T^*_2, or equivalently T_2 , in the case (15) with two further constraints on the parameters d_j, c_j and their relation with the operators C:

$$d_0d_1d_2d_3 = 1 \ , \ c_1^Nc_3^N = k^{2N}c_0^Nc_2^N \ , \quad C = \frac{c_1c_3}{k^2c_0c_2}, \tag{19}$$

[2]The convention we use here is in tune with the one in [6].

due to the connection of T_2^* with the following physical models.

(I) Discrete quantum pendulum. This is the situation under the constraint (19) with the following further ones,

$$D=1\ ;\quad \frac{d_0c_1d_3}{kc_2}U_2=\frac{kc_0}{d_1d_2c_3}U_4^{-1}(=:Q_{n-1})\ ,\quad \frac{c_1}{kc_0d_2d_3}U_1^{-1}=\frac{kd_0d_1c_2}{c_3}U_3(=:Q_n).$$

By (17), we have

$$Q_{n-1}Q_n=\frac{d_0c_1}{d_2c_3}\omega^{-1}V_1=\frac{d_0c_1}{d_2c_3}V_3^{-1}\ ,\quad Q_{n-1}Q_n^{-1}=\frac{c_0d_3}{c_2d_1}V_4=\frac{c_0d_3}{c_2d_1}\omega V_2^{-1}\ .$$

Then T_{2j}^*s in (16) become

$$\begin{aligned}\mathrm{T}_0^* &= \mathrm{T}_4^*=2\\ -\mathrm{T}_2^* &= 2(Q_n+Q_n^{-1}+Q_{n-1}+Q_{n-1}^{-1})\\ &\quad +k(qQ_{n-1}Q_n+q^{-1}Q_n^{-1}Q_{n-1}^{-1})+k^{-1}(qQ_nQ_{n-1}^{-1}+q^{-1}Q_{n-1}Q_n^{-1})\ .\end{aligned}\tag{20}$$

The above $-\mathrm{T}_2^*$ is the Hamiltonian of discrete quantum pendulum in [6] subject to the following evolution equation,

$$Q_{n+1}Q_n=(\frac{k+qQ_n}{1+qkQ_n})^2\ ,\quad Q_nQ_{n-1}=q^2Q_{n-1}Q_n\ .$$

(II) Discrete sine-Gordon (SG) Hamiltonian. This is the situation under the constraint (19) with one further identification,

$$\frac{c_1d_0d_3}{kc_2}D^{\frac{-1}{2}}U_2=\frac{kc_0}{c_3d_1d_2}D^{\frac{1}{2}}U_4^{-1}\quad .$$

In this case, we have

$$\begin{aligned}\mathrm{T}_0^* =&\quad D^{\frac{1}{2}}+D^{\frac{-1}{2}}\ ,\qquad\qquad \mathrm{T}_4^* = D^{\frac{-1}{2}}+D^{\frac{1}{2}}\ ,\\ -\mathrm{T}_2^* =&\quad \tfrac{kc_0d_2d_3}{c_1}D^{\frac{-1}{2}}U_1+\tfrac{c_1d_0d_3}{kc_2}D^{\frac{-1}{2}}U_2+\tfrac{kc_2d_0d_1}{c_3}D^{\frac{-1}{2}}U_3+\tfrac{c_3d_1d_2}{kc_0}D^{\frac{-1}{2}}U_4+\tfrac{c_1}{kc_0d_2d_3}D^{\frac{1}{2}}U_1^{-1}\\ &+\tfrac{kc_2}{c_1d_0d_3}D^{\frac{1}{2}}U_2^{-1}+\tfrac{c_3}{kc_2d_0d_1}D^{\frac{1}{2}}U_3^{-1}+\tfrac{kc_0}{c_3d_1d_2}D^{\frac{1}{2}}U_4^{-1}+\tfrac{kc_1d_0}{qc_3d_2}D^{\frac{-1}{2}}V_1+\tfrac{kqc_3d_2}{c_1d_0}D^{\frac{1}{2}}V_1^{-1}\\ &+\tfrac{c_0d_3}{kqc_2d_1}D^{\frac{-1}{2}}V_4+\tfrac{qc_2d_1}{kc_0d_3}D^{\frac{1}{2}}V_4^{-1}\ .\end{aligned}\tag{21}$$

The above $-\mathrm{T}_2^*$ can be identified with the discrete quantum sine-Gordon integral in [6], of which a detailed description will be given in the appendix of this paper.

3 The Polynomial T-Q Equation for Rational Degenerated Spectral Curve

In this section, we first derive the Baxter's T-Q polynomial relation for the general size L when the spectral curve $\mathcal{C}_{\vec{h}}$ is the rational degenerated one, by which we mean $\mathcal{C}_{\vec{h}}$ is a disjoint union of finite copies of the base x-rational curve. The formulation will be derived and reduced to a special degenerated case, of which the results were already obtained in the article [14]. Then we employ the formulae on the rational degenerated curve to the case L=4 incorporating the constraint (15) for the later discussion of this paper.

By the rational degenerated curve $\mathcal{C}_{\vec{h}}$, the condition of coordinates ξ_j^N is required to be constant independent of the variable x. The parameter h_js and the variables ξ_js of $\mathcal{C}_{\vec{h}}$ are subject to the constraints:

$$\frac{b_j^N d_j^N}{a_j^N c_j^N} = \frac{a_{j+1}^N b_{j+1}^N}{c_{j+1}^N d_{j+1}^N} , \quad \xi_j^{2N} = \frac{a_j^N b_j^N}{c_j^N d_j^N} \quad \text{for } \ 0 \leq j \leq L-1 .$$

In this situation, we define

$$r_j = \sqrt{\frac{b_{j-1} d_{j-1}}{a_{j-1} c_{j-1}}}, \quad j \in \mathbf{Z}_L . \tag{22}$$

Then $\mathcal{C}_{\vec{h}}$ contains the following $\tau_\pm$-invariant curve $\mathcal{C}$, upon which it suffices for us to formualte the Baxter's T-Q equation,

$$\mathcal{C} := \{(x, \xi_0, \ldots, \xi_{L-1}) \mid r_0^{-1}\xi_0 = \ldots = r_{L-1}^{-1}\xi_{L-1} = q^l \ , \ \ l \in \mathbf{Z}_N\} .$$

We shall make the identification of $\mathcal{C}$ with $\mathbf{P}^1 \times \mathbf{Z}_N$ via the following correspondence:

$$\mathcal{C} = \mathbf{P}^1 \times \mathbf{Z}_N \ , \quad (x, \ r_0 q^l, \ldots, r_{L-1} q^l) \longleftrightarrow (x, l) .$$

The automorphisms $\tau_\pm$ on $\mathcal{C}$ become

$$\tau_\pm : \ (x, l) \mapsto (q^{\pm 1} x, l-1) ,$$

by which the action (11) of $T(x)(:= T_{\vec{h}}(x))$ on $|x, l\rangle$ now takes the form,

$$T(x)|x, l\rangle = |q^{-1}x, l-1\rangle \Delta_-(x, l) + |qx, l-1\rangle \Delta_+(x, l) , \tag{23}$$

where $\Delta_\pm$ are the rational functions of x:

$$\begin{aligned} \Delta_-(x, l) \ &= d_0 \cdots d_{L-1} \prod_{j=0}^{L-1} (1 - x q^l d_j^{-1} c_j r_{j+1}) , \\ \Delta_+(x, l) \ &= d_0 \cdots d_{L-1} \prod_{j=0}^{L-1} \frac{1 - x^2 a_j^{-1} d_j^{-1} b_j c_j}{1 - x q^{-l} a_j^{-1} b_j r_{j+1}^{-1}} . \end{aligned}$$

With the substitutions,

$$(d_0 \cdots d_{L-1})^{-1} T(x) \mapsto T(x) \ , \ \ (d_0 \cdots d_{L-1})^{-1} \Delta_\pm(x, l) \mapsto \Delta_\pm(x, l) ,$$

The relation (23) still holds for the modified $\Delta_\pm$ with the expression,

$$\Delta_-(x, l) = \prod_{j=0}^{L-1} (1 - x c_j^* q^l) , \quad \Delta_+(x, l) = \prod_{j=0}^{L-1} \frac{1 - x^2 c_j^{*2}}{1 - x c_j^* q^{-l}}$$

where c_j^* is defined by

$$c_j^* = d_j^{-1} c_j r_{j+1} (= a_j^{-1} b_j r_{j+1}^{-1}) . \tag{24}$$

Furthermore, one can convert the expression (9) of the Baxter vacuum state over the elements of $\mathcal{C}$ to the following component-expression of the Baxter's vector $|x, l\rangle$:

$$\langle \mathbf{k} | x, l \rangle = q^{|\mathbf{k}|^2} \prod_{j=0}^{L-1} \frac{(x c_j^* q^{-l-2}; \omega^{-1})_{k_j}}{(x c_j^* q^{l+2}; \omega)_{k_j}} .$$

Here the bold letter $\mathbf{k}$ denotes a multi-index vector $\mathbf{k} = (k_0, \ldots, k_{L-1})$ for $k_j \in \mathbf{Z}_N$ with the square-length of $\mathbf{k}$ defined by $|\mathbf{k}|^2 := \sum_{j=0}^{L-1} k_j^2$. Each ratio-term in the above right hand side is given by a non-negative representative for each element in $\mathbf{Z}_N$ appeared in the formula. With the above descrption of $T(x)$ on the Baxter vacuum state $|x, l\rangle$, the discussion of Sect. 4 , also Proposition 2, 3 of Sect. 5 in [14] can be applied to our present situation. This enables us to state the following result on the Baxter's T-Q equation and its connection with the transfer matrix $T(x)$:

Theorem 1 *Let f^e, f^o be the functions on $\mathcal{C}$,*

$$f^e(x, 2n) := \prod_{j=0}^{L-1} \frac{(xc_j^*; \omega^{-1})_{n+1}}{(xc_j^*; \omega)_{n+1}}, \quad f^o(x, 2n+1) := \prod_{j=0}^{L-1} \frac{(xc_j^* q^{-1}; \omega^{-1})_{n+1}}{(xc_j^* q; \omega)_{n+1}} .$$

For $x \in \mathbf{P}^1$, $l \in \mathbf{Z}_N$, we define the following vectors in $\stackrel{L}{\otimes} \mathbf{C}^N$,

$$|x\rangle_l^e = \sum_{n=0}^{N-1} |x, 2n\rangle f^e(x, 2n)\omega^{ln} , \qquad |x\rangle_l^o = \textstyle\sum_{n=0}^{N-1} |x, 2n+1\rangle f^o(x, 2n+1)\omega^{ln} ,$$

$$|x\rangle_l^+ = |x\rangle_l^e q^{-l} u(qx) + |x\rangle_l^o u(x) \quad \text{where} \quad u(x) := \textstyle\prod_{j=0}^{L-1}(1 - x^N c_j^{*N})(xc_j^* q; q^2)_M .$$

Then

(i) $|x\rangle_l^e u(qx) = |x\rangle_l^o q^l u(x)$, or equivalently, $|x\rangle_l^+ = 2q^{-l}|x\rangle_l^e u(qx) = 2|x\rangle_l^o u(x)$.

(ii) The $T(x)$-transform on $|x\rangle_l^+$ is given by

$$q^{-l} T(x)|x\rangle_l^+ = |q^{-1}x\rangle_l^+ \Delta_-(x, -1) + |qx\rangle_l^+ \Delta_+(x, 0) , \quad l \in \mathbf{Z}_N .$$

(iii) For a common eigenvector $\langle\varphi|$ of $T(x)$ with the eigenvalue $\Lambda(x)$, the function $Q_l^+(x) (:= \langle\varphi|x\rangle_l^+)$ and $\Lambda(x)$ are polynomials with the properties:

$$\deg.Q_l^+(x) \leq (3M+1)L, \quad \deg.\Lambda(x) \leq 2[\tfrac{L}{2}], \quad \Lambda(x) = \Lambda(-x), \quad \Lambda(0) = q^{2l} + 1,$$

and the the following Baxter's T-Q equation holds:

$$q^{-l}\Lambda(x)Q_l^+(x) = \prod_{j=0}^{L-1}(1 - xc_j^* q^{-1})Q_l^+(xq^{-1}) + \prod_{j=0}^{L-1}(1 + xc_j^*)Q_l^+(xq) . \tag{25}$$

Furthermore for $0 \leq m \leq M$, $Q_m^+(x), Q_{N-m}^+(x)$ are elements in $x^m \prod_{j=0}^{L-1}(1 - x^N c_j^{*N})\, \mathbf{C}[x]$.

□

By (iii) of the above theorem, the equations (25) for the sector $m, N-m$ can be unified into a single one. For the rest of this paper the letter m will always denote an integer between 0 and M,

$$0 \leq m \leq M .$$

By introducing the polynomials $\Lambda_m(x), Q(x)$ via the relation,

$$(\Lambda_m(x),\ x^m \prod_{j=0}^{L-1}(1 - x^N c_j^{*N})Q(x)) = (q^{-m}\Lambda(x),\ Q_m^+(x)), \quad (q^m\Lambda(x),\ Q_{N-m}^+(x)) ,$$

the equations (25) for $l = m, N-m$, are equivalent to the following polynomial equation of $Q(x), \Lambda_m(x)$:

$$\Lambda_m(x)Q(x) = q^{-m}\prod_{j=0}^{L-1}(1 - xc_j^* q^{-1})Q(xq^{-1}) + q^m \prod_{j=0}^{L-1}(1 + xc_j^*)Q(xq) \ , \qquad (26)$$

with the constraints of $Q(x), \Lambda_m(x)$,

$$\text{deg.}Q(x) \leq ML - m, \quad \text{deg.}\Lambda_m(x) \leq 2[\frac{L}{2}], \quad \Lambda_m(x) = \Lambda_m(-x), \quad \Lambda_m(0) = q^m + q^{-m}.$$

By (14), the above $\Lambda_m(x)$ is indeed the eigenvalue of $T^*(x)$; while (26) corresponds the Baxter's T-Q equation (13) for $T^*(x)$ on the sectors $m, N-m$.

For L even, by the construction of $T^*_{\vec{h}}(x)$ one can see that the eiganvalues of T^*_L are non-zero, hence $\text{deg.}\Lambda_m(x) = L$. In certain situation, the equation (26) possess a solution with the reciprocal polynomials $Q(x), \Lambda_m(x)$. Here we call a polynomial $P(x)$ to be reciprocal if $P^\dagger(x) = P(x)$, where $P^\dagger(x)$ is the polynomial defined by

$$P^\dagger(x) := x^{\text{deg.}(P)}P(x^{-1}) \ .$$

The constraints of c_j^*s for the existence of a reciprocal polynomial solution of (26) will be related to the following criterion.

Proposition 1 *For L even, assume that the polynomial $\Lambda_m(x)$ in* (25) *is a reciprocal one, and the parameters c_j^*s and the degree d of the polynomial $Q(x)$ satisfy the conditions:*

(i) the collection of $c_0^*, \ldots, c_{L-1}^*$ is the same as that of $-c_0^{*-1}q, \ldots, -c_{L-1}^{*-1}q$, up to a permutation of indices.

(ii) $\prod_{j=0}^{L-1} c_j^* = q^{\frac{L}{2}}, \ q^{d+2m+\frac{L}{2}} = 1$.

Then $Q^\dagger(x)$ is also a solution of (25) *for $\Lambda_m(x)$.*

Proof. By substituting x by x^{-1} in (25), then multiplying x^{d+L} to the equation, one has the relation,

$$\Lambda_m(x)Q^\dagger(x) = q^{m+d}\prod_{j=0}^{L-1}c_j^*\prod_{j=0}^{L-1}(1 + xc_j^{*-1})Q^\dagger(xq^{-1}) + q^{-m-d-L}\prod_{j=0}^{L-1}c_j^*\prod_{j=0}^{L-1}(1 - xc_j^{*-1}q)Q^\dagger(xq) \ ,$$

By the condition (ii), we have

$$q^{m+d}\prod_{j=0}^{L-1}c_j^* = q^{-m} \ , \qquad q^{-m-d-L}\prod_{j=0}^{L-1}c_j^* = q^m \ .$$

With (i), the above equation of $Q^\dagger(x)$ is the same as (26). $\square$

The following algebraic fact was shown in [14] Lemma 6, which we just state here for later use in this paper.

Lemma 1 *Let n be an odd positive integer, A be a $n \times n$-matrix with complex entries $a_{i,j}$ satisfying the relations*

$$a_{i,j} = (-1)^{i+j+1}a_{n-j+1,n-i+1} \ , \quad \text{for } 1 \leq i,j \leq n \ .$$

Then A is a degenerated matrix.

$\square$

4 The Baxter's T-Q Polynomial Relation for L=4

For L=4, the parameter $\vec{h}$ subject to the constraint (15) which we will discuss in this paper for the rational degenerated case is confined only to the following situation:

$$qa_j = d_j = 1 \ , \quad -b_j = c_j = \begin{cases} k^{\frac{-1}{2}} & \text{for even } j, \\ k^{\frac{1}{2}} & \text{for odd } j, \end{cases} \tag{27}$$

in which case by (22), (24), we have $r_j = (-q)^{\frac{1}{2}}$ for all j , and

$$c_j^* = \begin{cases} (-q)^{\frac{1}{2}} k^{\frac{-1}{2}} & \text{for even } j, \\ (-q)^{\frac{1}{2}} k^{\frac{1}{2}} & \text{for odd } j. \end{cases} \tag{28}$$

The operators (16) in the transform matrix $T^*_{\vec{h}}(x)$ become

$$\begin{array}{rl} \mathrm{T}_0^* & = D^{\frac{1}{2}} + D^{\frac{-1}{2}}, \qquad\qquad \mathrm{T}_4^* = D^{\frac{1}{2}} C^{-1} + D^{\frac{-1}{2}} C \ , \\ -\mathrm{T}_2^* & = D^{\frac{-1}{2}} U_1 + D^{\frac{-1}{2}} U_2 + D^{\frac{-1}{2}} U_3 + D^{\frac{-1}{2}} U_4 + kq^{-1} D^{\frac{-1}{2}} V_1 + k^{-1} q^{-1} D^{\frac{-1}{2}} V_4 \\ & \quad + D^{\frac{1}{2}} U_1^{-1} + D^{\frac{1}{2}} U_2^{-1} + D^{\frac{1}{2}} U_3^{-1} + D^{\frac{1}{2}} U_4^{-1} + kq D^{\frac{1}{2}} V_1^{-1} + k^{-1} q D^{\frac{1}{2}} V_4^{-1} \ . \end{array}$$

By $C^N = 1$, $\Lambda_m(x)$ in (26) now takes the form

$$\Lambda_{m,l}(x) = (q^{m+l} + q^{-m-l}) x^4 + \lambda x^2 + q^m + q^{-m} \ , \quad 0 \le m \le M, \ \ 0 \le l \le 2M \ , \tag{29}$$

where λ is an eigenvalue of $-\mathrm{T}_2^*$ for the sectors $(m,l), (N-m,l)$, labelled by the values of $D^{\frac{\pm 1}{2}}, C^{\mp 1}$. The Baxter's T-Q equation (26) now takes the form,

$$\Lambda_{m,l}(x) Q(x) = q^{-m} \triangle(x(-q)^{\frac{-1}{2}}) Q(xq^{-1}) + q^m \triangle(x(-q)^{\frac{1}{2}}) Q(xq) \ , \tag{30}$$

with

$$\triangle(x) = (1 + 2cx + x^2)^2 \ , \quad c := \frac{1}{2}(k^{\frac{1}{2}} + k^{\frac{-1}{2}}) \ ,$$

and the solution $Q(x)$ is a polynomial of degree $\le 4M - m$. We are going to study the above equation (30) for the solutions $\lambda, Q(x)$ with a generic c. The $\Lambda_{m,l}(x), Q(x)$ will be called the eigenvalue and the eigen-polynomial of (30) when $Q(x)$ is a non-trivial function. In fact, the λ is an eigenvalue of $-\mathrm{T}_2^*$.

For the rest of this paper the parameter c will be assumed to be a generic complex number unless otherwise stated. The polynomial $Q(x)$ will always be denoted by

$$Q(x) = \sum_{j=0}^{d} \alpha_j x^j \ , \quad d := \deg . Q(x) \ ,$$

and we shall define $\alpha_j = 0$ for j not between 0 and d. An equivalent formulation of the equation (30) is the following system of difference equations in λ and α_js,

$$\nu_j \alpha_j + v_j \alpha_{j-1} + (\delta_j - \lambda) \alpha_{j-2} + u_j \alpha_{j-3} + \mu_j \alpha_{j-4} = 0 \ , \ \ j \ge 1 \ , \tag{31}$$

where the coefficients are defined by

$$\begin{array}{ll} \nu_j = q^{m+j} + q^{-m-j} - q^m - q^{-m}, & v_j = 4c\mathrm{i}(q^{m+j-\frac{1}{2}} - q^{-m-j+\frac{1}{2}}), \\ \delta_j = -(4c^2+2)(q^{m+j-1} + q^{-m-j+1}), & \\ u_j = -4c\mathrm{i}(q^{m+j-\frac{3}{2}} - q^{-m-j+\frac{3}{2}}), & \mu_j = q^{m+j-2} + q^{-m-j+2} - q^{m+l} - q^{-m-l} \ . \end{array} \tag{32}$$

Indeed, for the system (31) it suffices to consider those relations with the index j between 1 and $d+3$. Note that the relations in (31) for $2 \leq j \leq d+2$ give rise to the following eigenvalue problem,

$$\left\{ \begin{pmatrix} \delta_{d+2} & u_{d+2} & \mu_{d+2} & 0 & \cdots & 0 & 0 \\ v_{d+1} & \delta_{d+1} & u_{d+1} & \mu_{d+1} & \ddots & 0 & 0 \\ \nu_d & v_d & \delta_d & u_d & \mu_d & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \nu_4 & v_4 & \delta_4 & u_4 & \mu_4 \\ \vdots & \ddots & \ddots & \nu_3 & v_3 & \delta_3 & u_3 \\ 0 & \cdots & & 0 & \nu_2 & v_2 & \delta_2 \end{pmatrix} - \lambda \right\} \begin{pmatrix} \alpha_d \\ \alpha_{d-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_0 \end{pmatrix} = \vec{0} \ . \qquad (33)$$

Hence for a solution of the system (31), the λ can be regarded as an algebraic function of c, in which case the λ will take a value as c tends to some special element c_0.

Lemma 2 *For the equation* (30) *with a given* c *(no generic property required), the degree* d *of* $Q(x)$ *satisfies the following conditions,*

$$1 \leq d \leq 4M - m \ , \quad q^{d+2} = q^l \text{ or } q^{-2m-l} \ ,$$

and the zero multiplicity of $Q(x)$ *at the origin is one of* $0, N, N-2m, 2N-2m$.

Proof. The upper bound $4M - m$ of d is required by the assumption of (30). If $d = 0$, i.e., a non-zero constant is a solution $Q(x)$ of (30), we have

$$\Lambda_{m,l}(x) = q^{-m} \triangle(x(-q)^{\frac{-1}{2}}) + q^m \triangle(x(-q)^{\frac{1}{2}}) \ .$$

By the even function of $\Lambda_{m,l}(x)$, the above equality implies $q^{2m+1} = q^{2m+3} = 1$, hence $q^2 = 1$ which contradicts the odd property of N. Therefore $d \geq 1$. Comparing the coefficients of the highest degree of x in the equation (30), one has

$$q^{m+l} + q^{-m-l} = q^{-m-2-d} + q^{m+2+d} \ ,$$

which implies $q^{m+2+d} = q^{m+l}$ or q^{-m-l}, i.e., $q^{d+2} = q^l, q^{-2m-l}$. Denote r the zero multiplicity of $Q(x)$ at $x = 0$. By comparing the coefficients of degree r in the equality (30), we have

$$q^m + q^{-m} = q^{-m-r} + q^{m+r} \ ,$$

hence $r \equiv 0, -2m \pmod{N}$. Then the conclusion on r follows easily by the assumption $d \leq 4M - m$. □

Remark. For results obtained later in this paper on certain special cases, also in similar problems of the size $L = 3$ in [14], the solution $Q(x)$ in (26) usually possesses the property $Q(0) \neq 0$, in which case one can write

$$Q(x) = \prod_{j=1}^{d} (x - \frac{1}{z_j}) \ , \quad z_j \neq 0 \ .$$

For the equation (30), these z_js satisfy the relations:

$$q^{2m+2+d} \left(\frac{z_j^2 + 2\mathrm{i}cq^{\frac{1}{2}} z_j - q}{qz_j^2 - 2\mathrm{i}cq^{\frac{1}{2}} z_j - 1} \right)^2 = \prod_{n \neq j, n=1}^{d} \frac{z_n - qz_j}{qz_n - z_j} \ , \qquad j = 1, \ldots, d \ , \qquad (34)$$

which is the form of Bethe Ansatz appeared in some other literature, e.g. [8]. □

A special case with the above setting happens when the polynomial $\Lambda_{l,m}(x)$ in (30) is a reciprocal one. For the convenience, the Baxter's T-Q relation (30) will be called a symmetric T-Q polynomial relation if the following condition holds:

$$\Lambda^{\dagger}_{l,m}(x) = \Lambda_{l,m}(x), \quad \text{or equivalently} \quad q^l = 1, q^{-2m},$$

in which case, (29) becomes

$$\Lambda_{m,l}(x) = q^m + q^{-m} + \lambda x^2 + (q^m + q^{-m})x^4 \ , \tag{35}$$

and the coefficients (32) in the system (31) have the following form,

$$\begin{array}{ll} \nu_j = q^{m+j} + q^{-m-j} - q^m - q^{-m}, & v_j = 4c\mathrm{i}(q^{m+j-\frac{1}{2}} - q^{-m-j+\frac{1}{2}}) \\ \delta_j = -(4c^2+2)(q^{m+j-1} + q^{-m-j+1}), & \\ u_j = -4c\mathrm{i}(q^{m+j-\frac{3}{2}} - q^{-m-j+\frac{3}{2}}), & \mu_j = q^{m+j-2} + q^{-m-j+2} - q^m - q^{-m} \ . \end{array} \tag{36}$$

Note that by the equalities, $u_{j+1} = -v_j, \ \mu_{j+2} = \nu_j$, the transport of the square matrix in (33) is equal to the original one after substituting c by $-c$. Hence the eigenvalue λ for the symmetric polynomial T-Q relation necessarily becomes an algebraic function of c^2, equivalently, the following relation holds,

$$\lambda = \lambda(c) = \lambda(-c) \ . \tag{37}$$

Furthermore, the relation (30) is unchanged when substituting (c, x) by $(-c, -x)$; this implies that for a solution $Q(x; c)$ of (30), $Q(-x, ; -c)$ is also a solution.

We are going to determine the qualitative nature of the solution $Q(x)$ for a symmetric (30) polynomial equation. First we show the following lemma.

Lemma 3 *For the symmetric T-Q polynomial relation* (30)*, there is no non-trivial solution $Q(x)$ with the degree $d = N-2$ and the zero multiplicity at $x = 0$ equal to $N-2m$.*

Proof. Otherwise, one has $m \geq 1$ and

$$Q(x) = x^{N-2m}\widetilde{Q}(x) \ , \quad \widetilde{Q}(0) \neq 0 \ , \ \deg.\widetilde{Q} = 2m-2 \ . \tag{38}$$

Write

$$\widetilde{Q}(x) = \sum_{j=0}^{2m-2} \widetilde{\alpha}_j x^j \ .$$

Then $\widetilde{Q}(x)$ satisfies the relation

$$\Lambda_{m,l}(x)\widetilde{Q}(x) = q^m \triangle(x(-q)^{\frac{-1}{2}})\widetilde{Q}(xq^{-1}) + q^{-m}\triangle(x(-q)^{\frac{1}{2}})\widetilde{Q}(xq) \ , \tag{39}$$

or equivalently, the coefficients $\widetilde{\alpha}_j$s of $\widetilde{Q}(x)$ satisfy the system of equations,

$$\widetilde{\nu}_j\widetilde{\alpha}_j + \widetilde{v}_j\widetilde{\alpha}_{j-1} + (\widetilde{\delta}_j - \lambda)\widetilde{\alpha}_{j-2} + \widetilde{u}_j\widetilde{\alpha}_{j-3} + \widetilde{\mu}_j\widetilde{\alpha}_{j-4} = 0 \ , \quad 1 \leq j \leq 2m+1 \ , \tag{40}$$

where $\widetilde{\nu}_j, \widetilde{v}_j, \widetilde{\delta}_j, \widetilde{u}_j, \widetilde{\mu}_j$ are expressed by the similar forms as in (36) by changing m to $-m$ in the corresponding term. By $\widetilde{\nu}_1 \neq 0$, we have $m \geq 2$. By the equalities,

$$\widetilde{\nu}_j = \widetilde{\mu}_{2m+2-j}, \quad \widetilde{v}_j = \widetilde{u}_{2m+2-j}, \quad \widetilde{\delta}_j = \widetilde{\delta}_{2m+2-j} \ ,$$

$\widetilde{Q}^\dagger(x)$ also satisfies the equation (39). In general, for a polynomial $\widetilde{Q}(x)$ of degree $\tilde{d}$ satisfies (39), $\tilde{d} \equiv 2m-2, N-2 \pmod N$; the minimal possible $\tilde{d}$ is $2m-2$. Hence the solution space of $\widetilde{Q}(x)$ with degree $\leq 2m-2$ is of one-dimension. For the $\widetilde{Q}(x)$ in (38), $\widetilde{Q}^\dagger(x)$ is a scale-multiple of $Q(x)$, which implies $\widetilde{Q}^\dagger(x) = \pm\widetilde{Q}(x)$. Hence the polynomial $Q(x)$ is determined its coefficients $\widetilde{\alpha}_j$ with $0 \leq j \leq m-1$, which involve the equations in the range $1 \leq j \leq m+1$ in (40) subject to either one of the following two conditions: $\widetilde{\alpha}_j = \widetilde{\alpha}_{2m-2-j}$ for all j, or $\widetilde{\alpha}_j = -\widetilde{\alpha}_{2m-2-j}$ for all j. On the other hand, the relation (39) is the same when we substitute (c,x) by $(-c,-x)$, hence $\widetilde{Q}(-x,-c) = \widetilde{Q}(x,c)$, or equivalently, we may assume the coefficients $\widetilde{\alpha}_j = \widetilde{\alpha}_j(c)$ with the property,

$$\widetilde{\alpha}_0(c) = 1 \ , \quad \widetilde{\alpha}_j(-c) = (-1)^j \widetilde{\alpha}_j(c), \quad \text{for all } j \ . \tag{41}$$

For a solution of $\lambda, \widetilde{\alpha}_j$s for a general c in (40), λ is a solution of the eigenvalues problem arisen from the relations for $2 \leq j \leq m$, hence $\lambda = \lambda(c)$ as an algebraic function of c such that the limit of $\frac{\lambda(c)}{c^2}$, denoted by λ_∞, exists as $c \to \infty$. For $1 \leq j \leq m-1$, by $\widetilde{\nu}_j \neq 0$ one can conclude $\widetilde{\alpha}_j(c) = O(c^j)$ as $c \to \infty$. Denote

$$\begin{array}{ll} \mathsf{a}_j = \lim_{c\to\infty} \frac{\widetilde{\alpha}_j(c)}{c^j}, & 0 \leq j \leq m-1; \\ \widetilde{v}'_k = 4\mathrm{i}(q^{-m+k-\frac{1}{2}} - q^{m-k+\frac{1}{2}}), & \widetilde{u}'_k = -4\mathrm{i}(q^{-m+k-\frac{3}{2}} - q^{m-k+\frac{3}{2}}), \\ \widetilde{\delta}'_k = -4(q^{-m+k-1} + q^{m-k+1}), & 1 \leq k \leq m+1 \ . \end{array}$$

By multipling c^{-j} on (40), then considering the $c \to \infty$ limit of $1 \leq j \leq m+1$, we obtain the following matrix relation for λ_∞ and a_js,

$$\begin{pmatrix}
\widetilde{\delta}'_{m+1}-\lambda_\infty & 0 & 0 & \cdots & & & 0 \\
\widetilde{v}'_m & \widetilde{\delta}'_m-\lambda_\infty & 0 & 0 & \ddots & & \vdots \\
\widetilde{\nu}_{m-1} & \widetilde{v}'_{m-1} & \widetilde{\delta}'_{m-1}-\lambda_\infty & 0 & \ddots & \ddots & \vdots \\
0 & \ddots & \ddots & \ddots & & \ddots & \vdots \\
\ddots & \ddots & \ddots & & \ddots & \ddots & 0 \\
\vdots & \ddots & \widetilde{\nu}_4 & \widetilde{v}'_4 & \widetilde{\delta}'_4-\lambda_\infty & 0 & 0 \\
\vdots & \ddots & \ddots & \widetilde{\nu}_3 & \widetilde{v}'_3 & \widetilde{\delta}'_3-\lambda_\infty & 0 \\
0 & \cdots & \cdots & 0 & \widetilde{\nu}_2 & \widetilde{v}'_2 & \widetilde{\delta}'_2-\lambda_\infty \\
0 & \cdots & & \cdots & 0 & \widetilde{\nu}_1 & \widetilde{v}'_1
\end{pmatrix}
\begin{pmatrix} \mathsf{a}_{m-1} \\ \mathsf{a}_{m-2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \mathsf{a}_0 \end{pmatrix} = \vec{0} \ . \tag{42}$$

Note that $\mathsf{a}_0 = 1$, and $\widetilde{\delta}'_j \neq \widetilde{\delta}'_k$ for $2 \leq j \neq k \leq m+1$. The square matrix by deleting the last row in (42) becomes the eigenvalue problem with m-distinct eigenvalues, hence

$$\lambda_\infty = \widetilde{\delta}'_l \ , \text{ for some } 2 \leq l \leq m+1.$$

This implies $\mathsf{a}_k = 0$ for $l-2 < k \leq m-1$, and $\mathsf{a}_{l-2} \neq 0$. The relation on the row with $\widetilde{\delta}'_{l-1} - \lambda_\infty$ in (42) gives rise to the following relation of a_l and a_{l-1},

$$\widetilde{v}'_{l-1}\mathsf{a}_{l-2} + (\widetilde{\delta}'_{l-1} - \widetilde{\delta}'_l)\mathsf{a}_{l-3} = 0 \ . \tag{43}$$

One would expect the right lower square matrix of size $l-1$ with $\lambda_\infty = \widetilde{\delta}'_l$ has non-zero determinant, the statement of which is valid for a small number l by direct computation, hence it leads to a contradiction. However, it is a difficult task to obtain a mathematical proof of such statement for a general l. For our purpose, we are going to provide another way of justifying the conclusion of the lemma with the help of the relation (40). First we consider the case $l = m+1$, hence $\mathsf{a}_{m-1} \neq 0$, which implies $\widetilde{Q}^\dagger(x) = \widetilde{Q}(x)$. The relation (43) and the $(m+1)$-th equation in (40) become

$$\begin{array}{l} \widetilde{v}'_m \mathsf{a}_{m-1} + (\widetilde{\delta}'_m - \widetilde{\delta}'_{m+1}) \mathsf{a}_{m-2} = 0 \ , \\ (\widetilde{\delta}_{m+1} - \lambda(c))\widetilde{\alpha}_{m-1} + 2\widetilde{u}_{m+1}\widetilde{\alpha}_{m-2} + 2\widetilde{\mu}_{m+1}\widetilde{\alpha}_{m-3} = 0 \ . \end{array} \qquad (44)$$

By the property of λ in (37), the c-infinity limit of $c^{-(m-1)}$-multiple of the 2nd relation of (44) gives rise to the equality,

$$-4\mathsf{a}_{m-1} + 2\widetilde{u}'_{m+1}\mathsf{a}_{m-2} = 0 \ ,$$

which is incompatible with the first relation of (44). Now we may assume $2 \leq l \leq m$. By (41) and $\mathsf{a}_k = 0$ for $l-2 < k$, one has

$$\widetilde{\alpha}_k(c) = \mathsf{a}'_k c^{k-2} + \text{lower order term} \ , \quad \text{as } c \to \infty \ , \qquad l-1 \leq k \leq m-1.$$

By $\lambda_\infty = \widetilde{\delta}'_l$, one can take the c-infinity limit of $c^{-(l-2)}$-multiple of the l-th relation in (40), which yields the following identity:

$$\begin{array}{ll} \widetilde{\nu}_l \mathsf{a}'_l + \widetilde{v}'_l \mathsf{a}'_{l-1} + \frac{\widetilde{\delta}'_l}{2}\mathsf{a}_{l-2} + \widetilde{u}'_l \mathsf{a}_{l-3} = 0 \ , & \text{when } l \leq m-1; \\ \widetilde{\nu}_m \mathsf{a}_m + \widetilde{v}'_m \mathsf{a}'_{m-1} + \frac{\widetilde{\delta}'_m}{2}\mathsf{a}_{m-2} + \widetilde{u}'_m \mathsf{a}_{m-3} = 0 \ , & \text{when } l = m. \end{array} \qquad (45)$$

Similarly the c-infinity limit of $c^{-(l-1)}$-multiple of the $(l+1)$-relation gives rise to the following ones:

$$\begin{array}{ll} \widetilde{\nu}_{l+1}\mathsf{a}'_{l+1} + \widetilde{v}'_{l+1}\mathsf{a}'_l + (\widetilde{\delta}'_{l+1} - \widetilde{\delta}'_l)\mathsf{a}'_{l-1} + \widetilde{u}'_{l+1}\mathsf{a}_{l-2} = 0 \ , & \text{when } l \leq m-2; \\ \widetilde{v}'_m \mathsf{a}'_{m-1} + (\widetilde{\delta}'_m - \widetilde{\delta}'_{m-1})\mathsf{a}'_{m-2} + \widetilde{u}'_m \mathsf{a}_{m-3} = 0 \ , & \text{when } l = m-1; \\ (\widetilde{\delta}'_{m+1} - \widetilde{\delta}'_m)\mathsf{a}'_{m-1} + \widetilde{u}'_{m+1}(\mathsf{a}_{m-2} + \mathsf{a}_m) = 0 \ , & \text{when } l = m. \end{array} \qquad (46)$$

For $l = m$, by using $\mathsf{a}_m = \pm \mathsf{a}_{m-2}$ the relation (43) and the last one in (45), (46) will lead to a contradiction. For $l \leq m-2$, we continue the same procedure to the c-infinity limit of c^{-s}-multiple of the $(s+2)$-th relation for $s \geq l$, then obtain

$$\widetilde{\nu}_{s+2}\mathsf{a}'_{s+2} + \widetilde{v}'_{s+2}\mathsf{a}'_{s+1} + (\widetilde{\delta}'_{s+2} - \widetilde{\delta}'_s)\mathsf{a}'_s = 0 \ .$$

Hence one has the following relations for a'_ks,

$$\begin{pmatrix} \widetilde{\delta}'_{m+1} - \widetilde{\delta}'_l & 0 & 0 & \cdots & & 0 \\ \widetilde{v}'_m & \widetilde{\delta}'_m - \widetilde{\delta}'_l & 0 & 0 & \ddots & \vdots \\ \widetilde{\nu}_{m-1} & \widetilde{v}'_{m-1} & \widetilde{\delta}'_{m-1} - \widetilde{\delta}'_l & 0 & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & \vdots \\ \ddots & \ddots & \ddots & & \ddots & 0 \\ \vdots & \ddots & \widetilde{\nu}_{l+3} & \widetilde{v}'_{l+3} & \widetilde{\delta}'_{l+3} - \widetilde{\delta}'_l & 0 \\ \vdots & \ddots & \ddots & \widetilde{\nu}_{l+2} & \widetilde{v}'_{l+2} & \widetilde{\delta}'_{l+2} - \widetilde{\delta}'_l \end{pmatrix} \begin{pmatrix} \mathsf{a}'_{m-1} \\ \mathsf{a}'_{m-2} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \mathsf{a}'_l \end{pmatrix} = \vec{0},$$

which implies $\mathsf{a}'_s = 0$ for $s \geq l$. The relations (45) (46) become

$$\widetilde{v}'_l \mathsf{a}'_{l-1} + \frac{\widetilde{\delta}'_l}{2}\mathsf{a}_{l-2} + \widetilde{u}'_l \mathsf{a}_{l-3} = 0 \ , \quad (\widetilde{\delta}'_{l+1} - \widetilde{\delta}'_l)\mathsf{a}'_{l-1} + \widetilde{u}'_{l+1}\mathsf{a}_{l-2} = 0 \ ,$$

together with (43), this provides a contradiction to $\mathsf{a}_{l-2} \neq 0$. $\Box$

For the symmetric (30) equation, we are going to show the following property of the eigen-polynomial $Q(x)$.

Theorem 2 *For the symmetric T-Q polynomial relation* (30) *with a given reciprocal polynomial $\Lambda_{l,m}(x)$, assume that $Q(x)$ is a non-trivial polynomial solution. Then the solution space is 1-dimensional and it is generated by a monic polynomial $Q(x)$ of degree $2N-2-2m$ with $Q(0) \neq 0$ and $Q^\dagger(x) = \pm Q(x)$.*

Proof. By Lemma 2, we have $d = N-2, N-2m-2, 2N-2m-2$. First we are going to show that $d = 2N-2m-2$. Otherwise, d is one of odd integers, $N-2m-2$ or $N-2$. By Lemma 2 and 3, we may assume $Q(0) \neq 0$, and set $\alpha_0 = 1$. The coefficients α_js of $Q(x)$ satisfy the relation (31). When $d = N-2m-2$, we have $\nu_j \neq 0$ for $1 \leq j \leq d$, which implies the x-coefficients α_j of $Q(x)(= Q(x;c))$ are polynomials of c and $\lambda = \lambda(c)$, hence $\alpha_j = \alpha_j(c)$. As c tends to 0, the coefficients $\alpha_j = \alpha_j(0)$ of $Q(x;0)$ satisfy the corresponding relation (31),

$$\nu_j \alpha_j + (\delta_j - \lambda)\alpha_{j-2} + \mu_j \alpha_{j-4} = 0 \quad , \ 1 \leq j \leq d+3 \ ,$$

with $\alpha_1 = 0$. Hence $\alpha_j = 0$ for odd j, and the polynomial $Q(x;0)$ has an even degree $\leq N-2m-2$, which impossible by Lemma 2. It remains the case when $d = N-2$ with $Q(0;c) \neq 0$. Now the dimension of the $Q(x)$-solution space of (30) with deg.$Q(x) \leq N-2$ is equal to one. As $Q(-x,;-c)$ is also a solution of the symmetric T-Q relation, we have $Q(-x,;-c) = Q(x;c)$, equivalently to say, $\alpha_j(-c) = (-1)^j \alpha_j(c)$ for all j. Therefore $Q(x;0)$ is again a polynomial in x with an even degree $\leq N-2$, which contradicts to Lemma 2 for $c = 0$. Hence we have shown that any solution $Q(x)$ with the eigenvalue $\Lambda_{l,m}(x)$ must have the degree $d = 2N-2-2m$, which implies the dimension of the $Q(x)$-solution space equals to one. By (28), the conditions of Proposition 1 are satisfied for the polynomials $\Lambda_{l,m}(x)$, $Q(x)$, hence $Q^\dagger(x)$ is a also a solution of (30). Therefore $Q^\dagger(x) = \gamma Q(x)$ for some non-zero constant γ, which implies $\gamma^2 = 1$ and $Q(0) \neq 0$. Then the conclusion follows immediately. □

Remark. For a polynomial $Q(x)$ in the above proposition, the roots of $Q(x)$ are all non-zero; furthermore if x_k is a root, so is x_k^{-1}. Hence the collection of all roots x_k (counting multiplicity) is the same as that of x_k^{-1}s. The criterion for $Q^\dagger(x) = -Q(x)$ holds if and only if $Q(x)$ has the root $x = 1$ with a positive odd multiplicity. □

5 Solutions of Discrete Quantum Pendulum and Sine-Gordon Model in the Rational Degenerated case

In this section we are going to derive the complete solution of the symmetric T-Q polynomial relation (30); the sectors now are $(m,l) = (m,0), (m, N-2m)$. By Theorem 2, we may assume

$$d = 2N-2-2m \ , \quad Q^\dagger(x) = \pm Q(x).$$

Among the coefficients in (36), one has the following symmetric relations:

$$\nu_{d+4-j} = \mu_j, \quad v_{d+4-j} = u_j, \quad \delta_{d+4-j} = \delta_j \ .$$

The system (31) is equivalent to the eigenvalue problem (33) together with one more constraint

$$\nu_1 \alpha_1 + v_1 \alpha_0 = 0 \ , \tag{47}$$

where α_js satisfy either one of the following conditions

$$\begin{aligned} \alpha_i &= \alpha_{d-i} \quad \text{for } 0 \leq i \leq d, \quad \textit{i.e.}, \ Q^\dagger(x) = Q(x); && (48) \\ \alpha_i &= -\alpha_{d-i} \quad \text{for } 0 \leq i \leq d, \quad \textit{i.e.}, \ Q^\dagger(x) = -Q(x). && (49) \end{aligned}$$

Note that the polynomial $Q(x)$ is determined by only the half part of its coefficients: $\alpha_0, \ldots, \alpha_{\frac{d}{2}}$; while in the case (49), one has $\alpha_{\frac{d}{2}} = 0$. Furthermore through the transformations

$$\begin{array}{l} \nu_j, \upsilon_j, \delta_j, u_j, \mu_j \mapsto \mu_{j'}, u_{j'}, \delta_{j'}, \upsilon_{j'}, \nu_{j'} \ , \quad j' := d+4-j; \\ \alpha_i \mapsto \alpha_{d-i} \quad \text{or} \quad \alpha_i \mapsto -\alpha_{d-i} \ \text{ for } 0 \leq i \leq d \ , \end{array}$$

the equations for $j \geq \frac{d}{2}+3$ in (31) follows from those for $j \leq \frac{d}{2}+2$. So we need only to consider the relations for $1 \leq j \leq \frac{d}{2}+2$ in (31), which are regarded as the equations of λ and $\alpha_k, 0 \leq k \leq \frac{d}{2}$. Note that the $(\frac{d}{2}+2)$-th equation in (31) has the form

$$(\delta_{\frac{d}{2}+2} - \lambda)\alpha_{\frac{d}{2}} + u_{\frac{d}{2}+2}(\alpha_{\frac{d}{2}-1} + \alpha_{\frac{d}{2}+1}) + \mu_{\frac{d}{2}+2}(\alpha_{\frac{d}{2}-2} + \alpha_{\frac{d}{2}+2}) = 0 \ ; \tag{50}$$

which it is a trivial relation in the case (49).

By (19), for the rational degenerated case of discrete quantum pendulum and discrete sine-Gordon, it corresponds to $C = 1$, i.e., the sectors with $l = 0$ in the symmetric (30) relation; in particular, by (20) (27) the discrete quantum pendulum is given by $D = C = 1$, i.e., $(m, l) = (0, 0)$.

Theorem 3 *For the symmetric T-Q polynomial equation* (30)*, there are N distinct eigenvalues λ, each of which has 1-dimensional eigenspace generated by a monic eigen-polynomial $Q(x)$ of degree $d = 4M - 2m$ with $Q(0) \neq 0$ and $Q^\dagger(x) = \pm Q(x)$. Furthermore among these N eigen-polynomials, there are $M+1$ of $Q(x)$s with the type $Q^\dagger(x) = Q(x)$, and the other M ones are of the type $Q^\dagger(x) = -Q(x)$. In particular, the Baxter's T-Q polynomial relation of the* SG *model are those on the sectors $(m, l) = (m, 0)$, and the discrete quantum pendulum is the one for $(m, l) = (0, 0)$.*

Proof. The relation (47) is a non-trivial constraint for $0 \leq m \leq M-1$ by $\nu_1 \neq 0$; while for $m = M$, both ν_1 and υ_1 are zeros, hence (47) is a redundant one. In this proof, we shall first consider the case with $m = 0$, then $1 \leq m \leq M-1$, and finally on $m = M$.

(I) $m = 0$, i.e., $(m, l) = (0, 0)$, which is the rational degenerated case of discrete quantum pendulum. We have $d = 4M$. We consider the relations for $1 \leq j \leq \frac{d}{2}+2$ in the system (31) as equations of $\lambda, \alpha_0, ..., \alpha_{\frac{d}{2}}$. By $\nu_{\frac{d}{2}+1} = 0$, the problem is formulated in the following matrix form:

$$\begin{pmatrix} \upsilon_{\frac{d}{2}+1} & \delta_{\frac{d}{2}+1} - \lambda & u_{\frac{d}{2}+1} & \mu_{\frac{d}{2}+1} & 0 & \cdots & & 0 \\ \nu_{\frac{d}{2}} & \upsilon_{\frac{d}{2}} & \delta_{\frac{d}{2}} - \lambda & u_{\frac{d}{2}} & \mu_{\frac{d}{2}} & \ddots & & \vdots \\ 0 & \nu_{\frac{d}{2}-1} & \upsilon_{\frac{d}{2}-1} & \delta_{\frac{d}{2}-1} - \lambda & \ddots & \ddots & \ddots & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \nu_4 & \upsilon_4 & \delta_4 - \lambda & u_4 & \mu_4 \\ \vdots & \ddots & \ddots & \ddots & \nu_3 & \upsilon_3 & \delta_3 - \lambda & u_3 \\ 0 & \cdots & & \cdots & 0 & \nu_2 & \upsilon_2 & \delta_2 - \lambda \\ 0 & \cdots & \cdots & & \cdots & 0 & \nu_1 & \upsilon_1 \end{pmatrix} \begin{pmatrix} \alpha_{\frac{d}{2}} \\ \alpha_{\frac{d}{2}-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_0 \end{pmatrix} = \vec{0} \tag{51}$$

together with the constraint (50), which is now in the form

$$(q + q^{-1} - 2)(\alpha_{\frac{d}{2}+2} + \alpha_{\frac{d}{2}-2}) + 4c\mathrm{i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})(\alpha_{\frac{d}{2}+1} + \alpha_{\frac{d}{2}-1}) - (8c^2 + 4 + \lambda)\alpha_{\frac{d}{2}} = 0 \tag{52}$$

Note that the square matrix of size $N(=\frac{d}{2}+1)$ in (51) satisfies the condition of Lemma 1. Hence with $\nu_j \neq 0$ for $1 \leq j \leq \frac{d}{2}$, the system (51) has the one-dimensional eigenspace for any given c, λ with a basis element $(\alpha_k)_{0 \leq k \leq \frac{d}{2}}$ and $\alpha_0 = 1$. In fact for $1 \leq k \leq \frac{d}{2}$, α_k can be expressed by a polynomial of λ, c, which will be regarded as a polynomial in λ with coefficients in $\mathbf{C}[c]$, and denoted by $\alpha_k = p_k(\lambda)$. The λ-degree of α_k is given by deg. $p_k(\lambda) = [\frac{k}{2}]$. In the case (48), the relation (52) becomes

$$2(q+q^{-1}-2)p_{\frac{d}{2}-2}(\lambda) + 8c\mathrm{i}(q^{\frac{1}{2}} - q^{\frac{-1}{2}})p_{\frac{d}{2}-1}(\lambda) - (8c^2+4+\lambda)p_{\frac{d}{2}}(\lambda) = 0 \ ,$$

which defines λ as algebraic function of c, As the λ-degree of the above relation is equal to $M+1$, hence it gives rise to $M+1$ λ-values for a generic c. In the case (49), (52) is a trivial relation. The relation (51) becomes

$$\begin{pmatrix}
\delta_{\frac{d}{2}+1}-\lambda & u_{\frac{d}{2}+1} & \mu_{\frac{d}{2}+1} & 0 & \cdots & & 0 \\
v_{\frac{d}{2}} & \delta_{\frac{d}{2}}-\lambda & u_{\frac{d}{2}} & \mu_{\frac{d}{2}} & 0 & \cdots & 0 \\
\nu_{\frac{d}{2}-1} & v\frac{d}{2}-1 & \delta_{\frac{d}{2}-1}-\lambda & u_{\frac{d}{2}-1} & \mu_{\frac{d}{2}-1} & \ddots & 0 \\
0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\
\vdots & \ddots & \nu_4 & v_4 & \delta_4-\lambda & u_4 & \mu_4 \\
\vdots & \ddots & \ddots & \nu_3 & v_3 & \delta_3-\lambda & u_3 \\
0 & \cdots & & 0 & \nu_2 & v_2 & \delta_2-\lambda \\
0 & \cdots & & 0 & 0 & \nu_1 & v_1
\end{pmatrix}
\begin{pmatrix} \alpha_{\frac{d}{2}-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_0 \end{pmatrix} = \vec{0} \ .$$

The solution of the above system can be obtained from the system (51) alone, then by imposing the following constraint on λ,

$$\alpha_{\frac{d}{2}} = p_{\frac{d}{2}}(\lambda) = 0 \ .$$

As the λ-degree of p_k is equal to M, the above equation gives rise to M eigenvalues of λ, with the corresponding eigenvector having the components $\alpha_k = p_k(\lambda), 1 \leq k \leq \frac{d}{2}-1$. By Theorem 2, the $Q(x)$-eigenspaces are all of 1-dimension, hence the conclusion follows immediately.

(II) $1 \leq m \leq M-1$. We have $l = 0, N-2m$, and $d = 2N-2-2m$. By (36),

$$\nu_j = 0, \ 1 \leq j \leq \frac{d}{2}+1 \quad \Leftrightarrow \quad j = \mathbf{n} := N-2m \ . \tag{53}$$

In the case (48), the relations for $1 \leq j \leq \frac{d}{2}+2$ in the system (31) as equations of λ and $\alpha_k, 0 \leq k \leq \frac{d}{2}$ can be formulated in the following form,

$$\begin{pmatrix} S & T \\ 0 & U \end{pmatrix} \begin{pmatrix} \widetilde{\psi} \\ \psi \end{pmatrix} = \vec{0} \ , \qquad \psi := \begin{pmatrix} \alpha_{\mathbf{n}-1} \\ \vdots \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{pmatrix} \ , \quad \widetilde{\psi} := \begin{pmatrix} \alpha_{\frac{d}{2}} \\ \vdots \\ \vdots \\ \alpha_{\mathbf{n}+1} \\ \alpha_{\mathbf{n}} \end{pmatrix} \ , \tag{54}$$

together with the constraint (50). Here S, U are the square matrices of size $\frac{d}{2}-\mathbf{n}+1, \mathbf{n}$ and T is the $(\frac{d}{2}-\mathbf{n}+1)\times\mathbf{n}$ matrix with the following expressions:

$$S=\begin{pmatrix} v_{\frac{d}{2}+1} & \nu_{\frac{d}{2}+1}+\delta_{\frac{d}{2}+1}-\lambda & u_{\frac{d}{2}+1} & \mu_{\frac{d}{2}+1} & 0 & \cdots & 0 \\ \nu_{\frac{d}{2}} & v_{\frac{d}{2}} & \delta_{\frac{d}{2}}-\lambda & u_{\frac{d}{2}} & \mu_{\frac{d}{2}} & 0 & \vdots \\ 0 & \ddots & & & & & \\ \vdots & 0 & \ddots & & & & \vdots \\ \vdots & \cdots & \cdots & & & & \\ \vdots & \cdots & 0 & \nu_{\mathbf{n}+3} & v_{\mathbf{n}+3} & \delta_{\mathbf{n}+3}-\lambda & u_{\mathbf{n}+3} \\ 0 & \cdots & \cdots & 0 & \nu_{\mathbf{n}+2} & v_{\mathbf{n}+2} & \delta_{\mathbf{n}+2}-\lambda \\ 0 & \cdots & \cdots & & 0 & \nu_{\mathbf{n}+1} & v_{\mathbf{n}+1} \end{pmatrix},$$

$$U=\begin{pmatrix} v_{\mathbf{n}} & \delta_{\mathbf{n}}-\lambda & u_{\mathbf{n}} & \mu_{\mathbf{n}} & 0 & \cdots & 0 \\ \nu_{\mathbf{n}-1} & v_{\mathbf{n}-1} & \delta_{\mathbf{n}-1}-\lambda & u_{\mathbf{n}-1} & \mu_{\mathbf{n}-1} & 0 & \vdots \\ 0 & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \mu_4 \\ & \cdots & 0 & \nu_3 & v_3 & \delta_3-\lambda & u_3 \\ \vdots & \cdots & & 0 & \nu_2 & v_2 & \delta_2-\lambda \\ 0 & \cdots & & 0 & 0 & \nu_1 & v_1 \end{pmatrix},$$

$$T=\begin{pmatrix} 0 & \cdots & \cdots & & \cdots & 0 \\ \vdots & & \cdots & & & \vdots \\ 0 & \cdots & \cdots & & & \vdots \\ \mu_{\mathbf{n}+3} & 0 & \cdots & & & \vdots \\ u_{\mathbf{n}+2} & 0 & \cdots & & & 0 \\ \delta_{\mathbf{n}+1}-\lambda & u_{\mathbf{n}+1} & \mu_{\mathbf{n}+1} & 0 & \cdots & 0 \end{pmatrix}.$$

Note that there is the term $\nu_{\frac{d}{2}}$ in the second entry of the first row of S; while $\mu_{\mathbf{n}+2}=0$ in the matrix T. By (36), the matrix U satisfies the condition of Lemma 1 for any λ, hence by (53) the system $U\psi=0$ inside the system (54) has the one-dimensional solution generated by a vector ψ with

$$\alpha_0=1,\quad \alpha_k=p_k(\lambda)\in\mathbf{C}[c][\lambda],\quad \text{deg. } p_k(\lambda)=[\frac{k}{2}]\quad \text{for}\quad k<\mathbf{n}.$$

With the above vector ψ, we consider the system

$$S\widetilde{\psi}=-T\psi\ .$$

By (53), one can first solve α_k $(k>\mathbf{n})$ in terms of $\alpha_{\mathbf{n}},\lambda,c$ in the form

$$\alpha_k=r_k(\lambda)\alpha_{\mathbf{n}}+q_k(\lambda),\quad \text{where}\quad r_k(\lambda),q_k(\lambda)\in\mathbf{C}[c][\lambda]\ ,\quad \text{deg.}r_k(\lambda)+\tfrac{\mathbf{n}-1}{2}=\text{deg.}q_k(\lambda)=[\tfrac{k}{2}].$$

Furthermore $\alpha_{\mathbf{n}}$ satisfies the following relation,

$$0=v_{\frac{d}{2}+1}\alpha_{\frac{d}{2}}+(\nu_{\frac{d}{2}+1}+\delta_{\frac{d}{2}+1}-\lambda)\alpha_{\frac{d}{2}-1}+u_{\frac{d}{2}+1}\alpha_{\frac{d}{2}-2}+\mu_{\frac{d}{2}+1}\alpha_{\frac{d}{2}-3}=r(\lambda)\alpha_{\mathbf{n}}+q(\lambda)\ ,$$

where $r(\lambda),q(\lambda)\in\mathbf{C}[c][\lambda]$ with $\text{deg.}r(\lambda)+\frac{\mathbf{n}-1}{2}=\text{deg.}q(\lambda)=[\frac{d+2}{4}]$. Hence

$$\alpha_k=\frac{P_k(\lambda)}{r(\lambda)}\ ,\qquad P_k(\lambda):=r(\lambda)q_k(\lambda)-r_k(\lambda)q(\lambda)\ .$$

By multipling the α_k by $r(\lambda)$, we obtain a solution of (54) for all λ with the new $\alpha_k, 0 \leq k \leq \frac{d}{2}$ in the form

$$\alpha_k = P_k(\lambda) \ , \quad \deg.P_k(\lambda) = [\frac{k}{2}] + [\frac{d+2}{4}] - \frac{\mathbf{n}-1}{2} \ ; \tag{55}$$

in particular, $\deg.P_{\frac{d}{2}}(\lambda) = M$. Now the constraint (50) becomes

$$(\delta_{\frac{d}{2}+2} - \lambda)P_{\frac{d}{2}}(\lambda) + 2u_{\frac{d}{2}+2}P_{\frac{d}{2}-1}(\lambda) + 2\mu_{\frac{d}{2}+2}P_{\frac{d}{2}-2}(\lambda) = 0,$$

by which one can show that the above relation gives rise to $M+1$ λ-values for a generic c.

In the case (49), now (50) a trivial relation , we consider the following eiginvalue problem similar to the one in (54) by changing S to S^-,

$$\begin{pmatrix} S^- & T \\ 0 & U \end{pmatrix} \begin{pmatrix} \widetilde{\psi} \\ \psi \end{pmatrix} = \vec{0} \ , \qquad \psi := \begin{pmatrix} \alpha_{\mathbf{n}-1} \\ \vdots \\ \vdots \\ \alpha_1 \\ \alpha_0 \end{pmatrix} \ , \quad \widetilde{\psi} := \begin{pmatrix} \alpha_{\frac{d}{2}} \\ \vdots \\ \vdots \\ \alpha_{\mathbf{n}+1} \\ \alpha_{\mathbf{n}} \end{pmatrix} \ , \tag{56}$$

where the matrix S^- differs with S only on the $(1,2)$-th entry replacing $\nu_{\frac{d}{2}+1}$ by $-\nu_{\frac{d}{2}+1}$, i.e.,

$$S^- = \begin{pmatrix}
v_{\frac{d}{2}+1} & -\nu_{\frac{d}{2}+1} + \delta_{\frac{d}{2}+1} - \lambda & u_{\frac{d}{2}+1} & \mu_{\frac{d}{2}+1} & 0 & \cdots & 0 \\
\nu_{\frac{d}{2}} & v_{\frac{d}{2}} & \delta_{\frac{d}{2}} - \lambda & u_{\frac{d}{2}} & \mu_{\frac{d}{2}} & 0 & \vdots \\
0 & \ddots & & & & & \\
\vdots & 0 & \ddots & & & & \vdots \\
\vdots & \cdots & \cdots & & & & \\
\vdots & \cdots & 0 & \nu_{\mathbf{n}+3} & v_{\mathbf{n}+3} & \delta_{\mathbf{n}+3} - \lambda & u_{\mathbf{n}+3} \\
0 & \cdots & \cdots & 0 & \nu_{\mathbf{n}+2} & v_{\mathbf{n}+2} & \delta_{\mathbf{n}+2} - \lambda \\
0 & \cdots & \cdots & & 0 & \nu_{\mathbf{n}+1} & v_{\mathbf{n}+1}
\end{pmatrix} .$$

Then the problem (56) with the condition $\alpha_{\frac{d}{2}} = 0$ is equivalent to the solution for the case (49). As in the discussion of the eigenvalue problem (54), there is a solution α_js of (54) of the form $\alpha_k = P_k^-(\lambda)$ with the property in (55). Then $P_{\frac{d}{2}}^-$ is a λ-degree M polynomial and its zeros

$$\alpha_{\frac{d}{2}} = P_{\frac{d}{2}}^-(\lambda) = 0$$

give rise to M λ-values for the case (49). The conclusion now follows from Theorem 2.

(III) $m = M$. We have $l = 0, 1$, and $d = N - 1$. By the values $\nu_1, v_1, u_{N+2}, \mu_{N+2}$ in (32)are all zeros in this case, the relations of $j = 1, d+3$ in (31) are the redundant ones; hence the system (31) is equivalent to the eigenvalue problem (33) with $d = N - 1$. In the case (48), the collection of

$\alpha_k, 0 \le k \le \frac{d}{2}$, among the coefficients of $Q(x)$ is the solution of by the following eigenvalue problem,

$$\left\{ \begin{pmatrix} \delta_{M+2} & 2u_{M+2} & 2\mu_{M+2} & 0 & \cdots & 0 & 0 \\ v_{M+1} & \delta_{M+1}+\nu_{M+1} & u_{M+1} & \mu_{M+1} & \ddots & 0 & 0 \\ \nu_M & v_M & \delta_M & u_M & \mu_M & \ddots & 0 \\ 0 & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ \vdots & \ddots & \nu_4 & v_4 & \delta_4 & u_4 & \mu_4 \\ \vdots & \ddots & \ddots & \nu_3 & v_3 & \delta_3 & u_3 \\ 0 & \cdots & & 0 & \nu_2 & v_2 & \delta_2 \end{pmatrix} - \lambda \right\} \begin{pmatrix} \alpha_M \\ \alpha_{M-1} \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ \alpha_0 \end{pmatrix} = \vec{0} \ .$$

Note that the coefficients in the first and second rows have some extract terms comparing to the rest of entries. There are $M+1$ λ-eigenvalues for the above relation, which gives rise to the solution in the case (48). Then the rest M λ-values for (33) are those for the case (49). Then our conclusion follows from Theorem 2. $\Box$

Remark By Theorem 3, the eigen-polynomial $Q(x)$ in the symmetric T-Q polynomial equation (30) are in sectors $(m,l) = (m,0), (m, N-2m)$, and it has the form,

$$Q(x) = \prod_{j=1}^{2N-2-2m} (x - \frac{1}{z_j}) \ , \quad z_j \neq 0 \ .$$

The Bethe ansatz equation (34) for z_ks now becomes

$$(\frac{z_j^2 + 2\mathrm{i}cq^{\frac{1}{2}} z_j - q}{qz_j^2 - 2\mathrm{i}cq^{\frac{1}{2}} z_j - 1})^2 = \prod_{n\neq j, n=1}^{2N-2-2m} \frac{z_n - qz_j}{qz_n - z_j} \ , \qquad 1 \le j \le 2N-2-2m \ .$$

The solution $Q(x)$ in Theorem 3 imposes the reciprocal constraint on z_j's, i.e., $\{z_j\}_{j=1}^{2N-2-2m} = \{z_j^{-1}\}_{j=1}^{2N-2-2m}$ (counting the multiplicity). $\Box$

6 The General Spectral Curve for Discrete Quantum Pendulum and Discrete Sine-Gordon Model

In this section, we are going to explore the geometrical structure of the spectral curve $\mathcal{C}_{\vec{h}}$ (8) for the discrete quantum pendulum and SG model. By (15) (19), the parameter $\vec{h}$ for the curve $\mathcal{C}_{\vec{h}}$ has the following constraints,

$$\begin{array}{ll} a_j = q^{-1} d_j^{-1}, \quad b_j = -k^{-\epsilon_j} c_j^{-1}, & (\epsilon_j := (-1)^j), \ 0 \le j \le 3, \\ d_0 d_1 d_2 d_3 = 1, \quad c_1^N c_3^N = k^{2N} c_0^N c_2^N \ . & \end{array} \tag{57}$$

By the discussion in Sect. 8 of [14], the value ξ_j^Ns of the curve $\mathcal{C}_{\vec{h}}$ are determined by x^N, ξ_0^N, which will be denoted by $y := x^N, \eta := \xi_0^N$. The variables (y, η) satisfies the following equation of the curve $\mathcal{B}_{\vec{h}}$,

$$C_{\vec{h}}(y)\eta^2 + (A_{\vec{h}}(y) - D_{\vec{h}}(y))\eta - B_{\vec{h}}(y) = 0 \ ,$$

where $A_{\vec{h}}, B_{\vec{h}}, C_{\vec{h}}, D_{\vec{h}}$ are polynomials of y given by the relation,

$$\begin{pmatrix} -A_{\vec{h}}(y) & B_{\vec{h}}(y) \\ C_{\vec{h}}(y) & -D_{\vec{h}}(y) \end{pmatrix} := \prod_{j=0}^{3} \begin{pmatrix} -d_j^{-N} & -yc_j^{-N}k^{-\epsilon_j N} \\ yc_j^N & -d_j^N \end{pmatrix} .$$

In fact, by computation one obtains the explicit form of these polynomials,

$$\begin{array}{ll} A_{\vec{h}}(y) & = -(\delta^{-1} - y^2\gamma^{-1})(\delta - y^2\gamma) + y^2 k^{-N} c_0^{-N} d_0^{-N} c_2^N d_2^N (\delta + k^N\gamma)^2; \\ D_{\vec{h}}(y) & = -(\delta^{-1} - y^2\gamma^{-1})(\delta - y^2\gamma) + y^2 k^N c_0^N d_0^N c_2^{-N} d_2^{-N} (k^{-N}\delta^{-1} + \gamma^{-1})^2; \\ B_{\vec{h}}(y) & = y(\delta^{-1} - y^2\gamma^{-1})\{k^{-N} c_0^{-N} d_0^{-N}(\delta + k^N\gamma) + c_2^{-N} d_2^{-N}(k^{-N}\delta^{-1} + \gamma^{-1})\}; \\ C_{\vec{h}}(y) & = -y(\delta - y^2\gamma)\{k^N c_0^N d_0^N (k^{-N}\delta^{-1} + \gamma^{-1}) + c_2^N d_2^N (\delta + k^N\gamma)\} , \end{array}$$

where δ, γ are defined by

$$\delta := d_0^N d_1^N = d_2^{-N} d_3^{-N} , \quad \gamma := \frac{c_0^N k^N}{c_1^N} = \frac{c_3^N}{k^N c_2^N} .$$

Eliminating the y-factor in the equation of $\mathcal{B}_{\vec{h}}$, we obtain an irreducible curve. For the notional convenience, we still denote the curve by $\mathcal{B}_{\vec{h}}$ again, now with the equation,

$$\mathcal{B}_{\vec{h}} : \ \mathsf{a}(y^2\gamma - \delta)\eta^2 + \mathsf{b}y\eta + \mathsf{c}(y^2\gamma^{-1} - \delta^{-1}) = 0 , \tag{58}$$

where $\mathsf{a}, \mathsf{b}, \mathsf{c}$ are the parameters defined by

$$\begin{array}{ll} \mathsf{a} := & k^N c_0^N d_0^N (k^{-N}\delta^{-1} + \gamma^{-1}) + c_2^N d_2^N (\delta + k^N\gamma) , \\ \mathsf{b} := & k^{-N} c_0^{-N} d_0^{-N} c_2^N d_2^N (\delta + k^N\gamma)^2 - k^N c_0^N d_0^N c_2^{-N} d_2^{-N} (k^{-N}\delta^{-1} + \gamma^{-1})^2 , \\ \mathsf{c} := & k^{-N} c_0^{-N} d_0^{-N} (\delta + k^N\gamma) + c_2^{-N} d_2^{-N} (k^{-N}\delta^{-1} + \gamma^{-1}). \end{array}$$

The curves $\mathcal{B}_{\vec{h}}$ form a family of elliptic curves depending on the 4 parameters, $\delta, \gamma, k^N c_0^N d_0^N, c_2^N d_2^N$. The curve $\mathcal{C}_{\vec{h}}$ is a $\mathbf{Z}_N^5$ (branched) cover over $\mathcal{B}_{\vec{h}}$, whose covering transformation group contains the $\tau_\pm$ in (18). For a generic data $\vec{h}$, $\mathcal{C}_{\vec{h}}$ is a high genus curve; indeed the genus is equal to $2N^3(N-1)(N+2)+1$.

Now we make a qualitative analysis of the solutions of the Baxter's T-Q relation (13) in the problem of the discrete quantum pendulum (20) and SG model (21). All the linear transformations appeared in the expressions of (16) are operators of the vector space $\stackrel{4}{\otimes} \mathbf{C}^{N*}$ As D, C and U_js in the expression of $T^*_{\vec{h}}(x)$ are commuting operators, by $(\frac{k^2 c_0 c_2}{c_1 c_3} C)^N = 1$ the eigenvalue of $T^*_{\vec{h}}(x)$ are still of the form (29) with λ depending on k; while for the discrete quantum pendulum and SG model, it becomes (35). By the expressions of D and C, it is not hard to see that the common eigen-subspaces of $\stackrel{4}{\otimes} \mathbf{C}^{N*}$ for the commuting operators $D^{\frac{1}{2}}, \frac{k^2 c_0 c_2}{c_1 c_3} C$ are all of dimension N^2. The eigenspace decomposition of $\stackrel{4}{\bigotimes} \mathbf{C}^{N*}$ is denoted by

$$\bigotimes^4 \mathbf{C}^{N*} = \bigoplus_{n,n' \in \mathbf{Z}_N} \mathbf{E}_{n,n'} , \quad \mathbf{E}_{n,n'} \simeq \mathbf{C}^{N^2} ,$$

where $D^{\frac{1}{2}}, \frac{c_1 c_3}{k^2 c_0 c_2} C^{-1}$ act on $\mathbf{E}_{n,n'}$ by the multiplication of $q^n, q^{n'}$ respectively. By the relations of U_j's and C, D in (17), each $\mathbf{E}_{n,n'}$ is stable under U_js. The operators U_js on $\mathbf{E}_{n,n'}$ are determined only by those of U_1, U_2 which form the Weyl algebra (1), i.e., the following relations hold:

$$U_2 U_1 = \omega U_1 U_2 , \quad U_1^N = U_2^N = 1 .$$

This implies the operator T_2^* in (16) is determined by the representation $\mathbf{E}_{n,n'}$ of the Weyl algebra for each sector labelled by the values of $\mathrm{T}_0^*, \mathrm{T}_4^*$ corresponding to (n,n'). As the irreducible representation of the Weyl algebra is uniquely given by the standard one on $\mathbf{C}^N$, $\mathbf{E}_{n,n'}$ is isomorphic to the sum of N-copies of the standard one as the Weyl algebra modules. In particular, the eigenvalues of $-\mathrm{T}_2^*$ on the vector space $\mathbf{E}_{n,n'}$ are induced from the standard representation of the Weyl algebra ; each eigenvalue gives rise to N eigenvectors in $\mathbf{E}_{n,n'}$. By (9) (57), the Baxter vacuum state $|p\rangle \in \overset{4}{\otimes}\ \mathbf{C}^N$ for $p \in \mathcal{C}_{\vec{h}}$ is now defined by $|p\rangle = |p_0\rangle \otimes |p_1\rangle \otimes |p_2\rangle \otimes |p_3\rangle$ with the vector $|p_j\rangle$ in $\mathbf{C}^N$ given by

$$\langle 0|p_j\rangle = 1 \ , \qquad \frac{\langle m|p_j\rangle}{\langle m-1|p_j\rangle} = \frac{\xi_{j+1}k^{\epsilon_j}c_jq^{2m-1} + xd_j}{-\xi_j(\xi_{j+1}xc_jq^{2m} - d_j)k^{\epsilon_j}c_jd_j} \ .$$

For a generic $\vec{h}$, the evaluation of vectors of $\overset{4}{\otimes}\ \mathbf{C}^{*N}$ on the Baxter vacuum state, $* \mapsto \langle *|p\rangle$, induces an isomorphism between $\overset{4}{\otimes}\ \mathbf{C}^{*N}$ and a N^4-dimensional subspace of rational functions of $\mathcal{C}_{\vec{h}}$, in which $\mathbf{E}_{n,n'}$ gives rise a functional space of $\mathcal{C}_{\vec{h}}$ with the dimension N^2, denoted by $\epsilon(\mathbf{E}_{n,n'})$, with the Weyl algebra module structure induced from that of $\mathbf{E}_{n,n'}$. In the Baxter's T-Q equation (13) on $\mathcal{C}_{\vec{h}}$ with $\Lambda^*(x) = \Lambda_{m,l}(x)$ in (29), the function $Q(p)$ is the eigenfunction of $T^*_{\vec{h}}(x)$ in $\epsilon(\mathbf{E}_{n,n'})$ for $(n,n') = (m,l), (N-m, N-l)$, with the multiplicity N. By (20) (21), the Baxter's T-Q relation for the discrete quantum pendulum and SG model is the one with $\Lambda^*(x)$ being the reciprocal polynomial (35). To determine these eigen-functions $Q(p)$ would require the understanding of its zeros and poles from the expression of the Baxter vacuum state, which is a difficult task at this moment. The possible role of the elliptic function theory of $\mathcal{B}_{\vec{h}}$ in the solutions of Baxter's T-Q relation on $\mathcal{C}_{\vec{h}}$, hence further understanding on the eigenvalue problem of the models (20) (21), would be the core of our future work in this aspect.

7 Conclusions and Perspectives

We have studied the discrete quantum pendulum and discrete sine-Gordon model in the frame work of the quantum inverse scattering method. The eigenvalues and eigenvectors problem is governed by the Baxter's T-Q relation which arises from the Baxter vacuum state on the spectral curve using the general scheme of diagonalizating the transfer matrix of a fixed finite size L in [14]. We have demonstrated the role of algebraic geometry in the qualitative study of the T-Q relation for L=3 in [14], and L=4 now in this article, which both have an intimate relationship with integrable Hamiltonian spin-chains of physical interest. The spectral curves depend on parameters encoded in the expression of the Hamiltonian system. For a generic parameter, the curve over which the Baxter's T-Q relation is formulated has a high genus as demonstrated in Sect. 6. However both the case $L = 3, 4$, the spectral curves have thus far presented a common feature as branched covers over elliptic curves. One might expect to employ the elliptic function theory to solutions of those Baxter's T-Q relation to enrich our understanding the corresponding Hamiltonian spectrum problem. This would be a challenging program on which we hope to make progress in future.

When the spectral curve degenerates into rational curves where the geometry play little role, by using the data obtained in [14] for some special degenerated curves, we derive the polynomial form of the Baxter's T-Q relation for a finite chain system of arbitrary size L in Sect. 3. We apply these results to the case L=4 in Sect. 4 with parameters in the setting of the discrete quantum pendulum and sine-Gordon model. In this case, an extra symmetry has naturally been imposed on the T-Q polynomial equation, indeed it is governed by the reciprocal property of the equation. With this constraint, we present a detailed and rigorous mathematical derivation of solutions of the Baxter's T-Q polynomial equation in Theorem 3. Surprisingly the conclusion on these polynomial solutions has been much in tune with the one for $L = 3$ obtained in [14] on the study of Hofstadter type model (see Theorem 3 of the article there). Furthermore the exact connection of the Baxter's T-Q polynomial equation with Bethe Ansatz in other literature has been clearified in all these cases. The results obtained in this paper signal some further mathematical feature of the Baxter's T-Q polynomial equation, namely novel equivalence with the theory of q-Strum-Liouville problem at roots of unity $q^N = 1$, a view in accordance with a recently-observed connection between Bethe Ansatz of XXZ model and q-Strum-Liouville type relation in [12]. The facts discovered in all these work could be served to demonstrate that a systematically mathematical theory embodied in the Baxter's T-Q polynomial equation (or algebraic Bethe Ansatz) would emerge in the study of q-difference operators. Accordingly, the relationship along this line is now under our consideration with progress now being made.

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Appendix: Discrete Quantum Sine-Gordon Hamitonian

For the consistency of our notion of discrete quantum sine-Gordon Hamiltonian with the ones used in other literature, we make an identification of the discrete sine-Gordon integral in [6] with the T_j^*s of (21) in this paper. In Sect. 5 of [6], the discrete quantum sine-Gordon Hamiltonian arises from the following commuting operators[3],

$$\begin{aligned} A^{(0)} &= \mathsf{U}_2^{-1}\mathsf{U}_1^{-1}, \\ A^{(1)} &= \mathsf{U}_2\mathsf{Z}_2\mathsf{U}_1^{-1} + \mathsf{U}_1\mathsf{Z}_1\mathsf{U}_2^{-1} + \mathsf{V}_2\mathsf{h}_2\mathsf{h}_1^*\mathsf{V}_1^{-1} , \\ A^{(2)} &= \mathsf{U}_2\mathsf{Z}_2\mathsf{U}_1\mathsf{Z}_1, \end{aligned}$$

where the lower index $j = 1, 2$ indicates the site of operators in the same algebra generated by $\mathsf{U}, \mathsf{V}, \mathsf{Z}$, subject to the relation $\mathsf{UV} = \mathsf{q}^{\frac{-1}{2}}\mathsf{VU}$ with Z the central element, and h is defined by $\mathsf{h} = k^{\frac{-1}{2}} + k^{\frac{1}{2}}\mathsf{q}^{\frac{-1}{2}}\mathsf{U}^2\mathsf{Z}$. The sine-Gordon ($SG$) integral is defined by the operator

$$\widetilde{H} = A^{(1)} + A^{(1)*} .$$

One can show that

$$\mathsf{V}_2\mathsf{h}_2\mathsf{h}_1^*\mathsf{V}_1^{-1} = \mathsf{q}^{\frac{-1}{2}}\mathsf{V}_2\mathsf{U}_2^2\mathsf{Z}_2\mathsf{V}_1^{-1} + \mathsf{q}^{\frac{1}{2}}\mathsf{V}_2\mathsf{Z}_1^{-1}\mathsf{U}_1^{-2}\mathsf{V}_1^{-1} + k\mathsf{V}_2\mathsf{U}_2^2\mathsf{Z}_2\mathsf{Z}_1^{-1}\mathsf{U}_1^{-2}\mathsf{V}_1^{-1} + k^{-1}\mathsf{V}_2\mathsf{V}_1^{-1} .$$

For the convenience in expressing $\widetilde{H}$, we denote

$$\begin{aligned} \mathsf{W}_1 &:= \mathsf{q}^{\frac{1}{2}}\mathsf{V}_2\mathsf{Z}_1^{-1}\mathsf{U}_1^{-2}\mathsf{V}_1^{-1}, \quad & \mathsf{W}_2 &:= \mathsf{U}_1\mathsf{Z}_1\mathsf{U}_2^{-1} , \\ \mathsf{W}_3 &:= \mathsf{q}^{\frac{-1}{2}}\mathsf{V}_2\mathsf{U}_2^2\mathsf{Z}_2\mathsf{V}_1^{-1} , \quad & \mathsf{W}_4 &:= \mathsf{U}_2\mathsf{Z}_2\mathsf{U}_1^{-1} . \end{aligned}$$

Note that

$$\mathsf{W}_2\mathsf{W}_4 = A^{(0)}A^{(2)} . \tag{59}$$

[3] Here we use the sans serif type style, instead of the italic type style in [6], for operators appeared in the right hand side of the expressions for the purpose of less confusion with notations used in this paper.

We have

$$k\mathsf{V}_2\mathsf{U}_2^2\mathsf{Z}_2\mathsf{Z}_1^{-1}\mathsf{U}_1^{-2}\mathsf{V}_1^{-1} = k\mathsf{q}^{\frac{-1}{2}}A^{(0)}\mathsf{W}_3\mathsf{W}_2^{-1}, \quad k^{-1}\mathsf{V}_2\mathsf{V}_1^{-1} = k^{-1}\mathsf{q}^{\frac{-1}{2}}A^{(2)}\mathsf{W}_4^{-1}\mathsf{W}_1,$$

hence

$$A^{(1)} = \mathsf{W}_1 + \mathsf{W}_2 + \mathsf{W}_3 + \mathsf{W}_4 + k\mathsf{q}^{\frac{-1}{2}}A^{(0)}\mathsf{W}_3\mathsf{W}_2^{-1} + k^{-1}\mathsf{q}^{\frac{-1}{2}}A^{(2)}\mathsf{W}_4^{-1}\mathsf{W}_1 \ .$$

With $\mathsf{q}^{\frac{1}{2}} = q$ and the following identification of the above operators appeared in $A^{(j)}$s and those in T^*_{2j}s under the condition $d_0d_1d_2d_3 = 1$,

$$\begin{array}{lllllll} A^{(0)} & \leftrightarrow & D^{\frac{1}{2}} \ ; & & A^{(2)} & \leftrightarrow & \frac{c_1c_3}{k^2c_0c_2}D^{\frac{1}{2}}C^{-1}; \\ \mathsf{W}_1 & \leftrightarrow & \frac{kc_0d_2d_3}{c_1}D^{\frac{-1}{2}}U_1; & & \mathsf{W}_2 & \leftrightarrow & \frac{kc_2}{c_1d_0d_3}D^{\frac{1}{2}}U_2^{-1}; \\ \mathsf{W}_3 & \leftrightarrow & \frac{kc_2d_0d_1}{c_3}D^{\frac{-1}{2}}U_3; & & \mathsf{W}_4 & \leftrightarrow & \frac{c_3d_1d_2}{kc_0}D^{\frac{-1}{2}}U_4, \end{array}$$

then imposing further constraints,

$$\mathsf{W}_4^{-1} \ \leftrightarrow \ \frac{c_1d_0d_3}{kc_2}D^{\frac{-1}{2}}U_2 \ , \quad A^{(2)} \ \leftrightarrow \ D^{\frac{1}{2}} \ , \tag{60}$$

by the equalities $V_1 = U_3U_2, V_4 = U_2U_1$ in (17), the SG-integral $\widetilde{H}$ becomes $-\mathrm{T}^*_2$ in (21). Then (60) gives rise to the identification,

$$\mathsf{W}_2 = \mathsf{W}_4 \ , \quad A^{(0)} = A^{(2)} \ ,$$

or equivalently,

$$\frac{kc_2}{c_1d_0d_3}D^{\frac{1}{2}}U_2^{-1} = \frac{c_3d_1d_2}{kc_0}D^{\frac{-1}{2}}U_4 \ , \qquad \frac{c_1c_3}{k^2c_0c_2} = C \ .$$

Note that the above relations are consistent with the relation $U_2U_4 = C^{-1}D$ in (17). Hence we obtain the relations (19) (21).

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